Almost Quantum Theory

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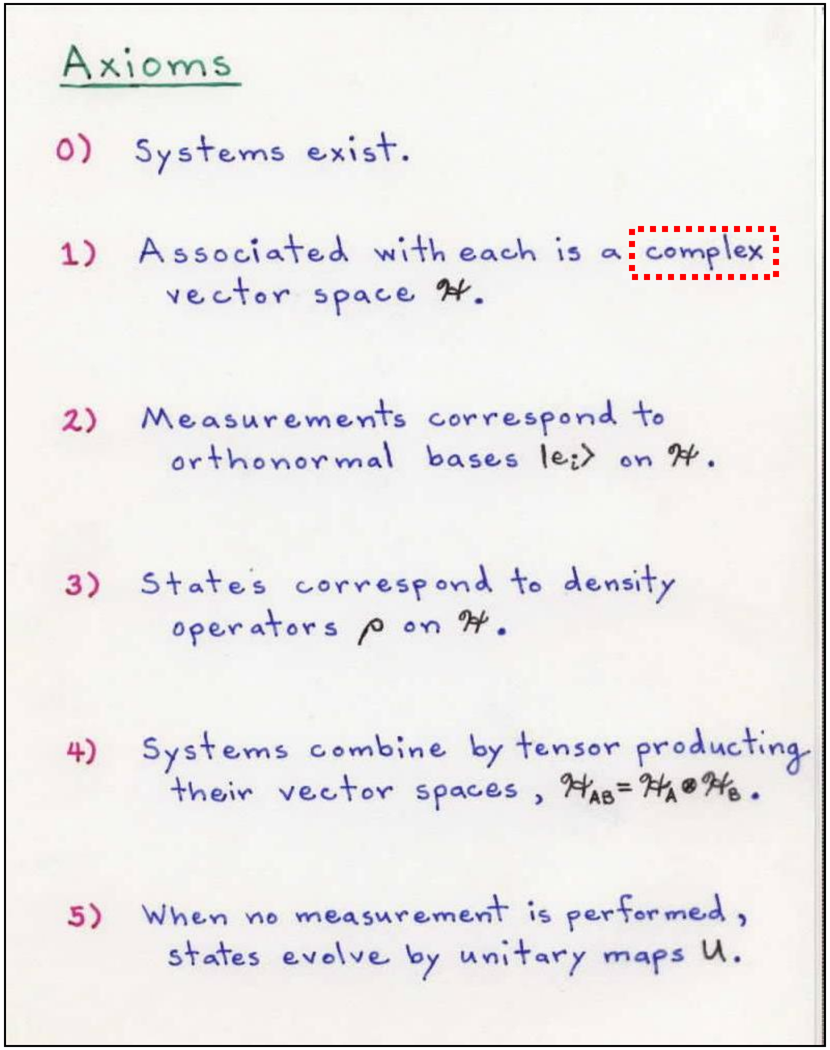


Axioms

0) Systems exist.

1) Associated with each is a complex vector space $\mathcal{H}$.

2) Measurements correspond to orthonormal bases $|e_i\rangle$ on $\mathcal{H}$.

3) States correspond to density operators ρ on $\mathcal{H}$.

4) Systems combine by tensor producting their vector spaces, $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

5) When no measurement is performed, states evolve by unitary maps U.

Chris Fuchs, "*The Oyster and the Quantum*"

- Where does the elaborate mathematical structure of quantum theory "come from"?
- How would quantum theory change if we modified the axioms?
- What is the role of complex numbers in quantum theory?
- Can we develop a "toy model" of quantum theory that is much simpler but has many of the same general features?

interference
entanglement
non-classical computation
....

An imaginary world

A world without probability

Probability: Events have probabilities

$$0 \leq p(x) \leq 1$$

normalization: $\displaystyle\sum_x p(x) = 1$

In N » 1 trials, event x occurs N_x times.

With high probability, $p(x) \approx \dfrac{N_x}{N}$

Naive connection:

$$x \in P \Leftrightarrow p(x) \neq 0$$

Possibility: Some events are possible

$$P = \{x, x', \ldots\}$$

normalization: $P \neq \emptyset$

In N trials, the set of events that occur is $R \subseteq P$

Modal logic explores the ideas of possibility and necessity (propositions p, $\Diamond p$, $\Box p = \sim\Diamond(\sim p)$, etc.).

Actual Quantum Theory (AQT)

States

Hilbert space H over field C

Pure state is vector $|\psi\rangle$

$$\langle\psi|\psi\rangle = 1$$

Measurement

Orthonormal basis $\{|k\rangle\}$ for H

$$|\psi\rangle = \sum_k \psi_k |k\rangle$$

Probability: $P(k) = |\psi_k|^2$

Time evolution

$|\psi\rangle \longrightarrow U|\psi\rangle$ (U unitary)

Composite systems

$$H^{12} = H^1 \otimes H^2$$

Modal Quantum Theory (MQT)

States

Vector space V over field F

Pure state is vector $|\alpha)$

$$|\alpha) \neq 0$$

Measurement

Basis $\{|k)\}$ for V

$$|\alpha) = \sum_k \alpha_k |k)$$

Possibility: $\alpha_k \neq 0$

Time evolution

$|\alpha) \longrightarrow T|\alpha)$ (T invertible)

Composite systems

$$V^{12} = V^1 \otimes V^2$$

"Mobits"

$Z_2 = \{0,1\}$
$1+1 = 0$

Simplest possible situation: $F = Z_2$ and dim $V = 2$

Only three possible states $|0)\,,|1)\,,|\sigma) = |0) + |1)$

Three basis sets (X, Y, Z)

$$\boxed{\begin{aligned} |0_x) &= |1) \\ |1_x) &= |\sigma) \end{aligned}} \quad \boxed{\begin{aligned} |0_y) &= |\sigma) \\ |1_y) &= |0) \end{aligned}} \quad \boxed{\begin{aligned} |0_z) &= |0) \\ |1_z) &= |1) \end{aligned}}$$

A cautionary tale: Given MQT state $|\sigma)$

Measurement basis includes $|0)$

Is this result possible?

Y basis: $|\sigma) = |0_y)$

1_y not possible

Z basis: $|\sigma) = |0_z) + |1_z)$

0_z possible

The dual view

A better way: Think about the dual space V^*.

$$\{|a)\} \leftrightarrow \{(a|\}$$

V basis $\qquad$ V^* basis

$$(a\,|\phi\,) = \phi_a$$

in $\;|\phi) = \sum_a \phi_a\,|a)$

NB: Correspondence $|a) \leftrightarrow (a|$ depends on the *entire* basis.

- A measurement is associated with a basis for V^*. (This always corresponds to a basis for V itself.)
- Each measurement result a is represented by a dual vector $(\,a\,|$ -- the "effect functional".
- Possibility rule: a is possible iff $(a\,|\phi\,) \neq 0$

  This depends only on the state and the effect functional!
- Vectors $|\phi)$ and $c\,|\phi)$ are equivalent ("same state").

The dual view

A better way: Think about the dual space *V**.

$$\{|a)\} \leftrightarrow \{(a|\}$$

V basis

*V** basis

$$(a\,|\phi\,) = \phi_a$$

in $|\phi) = \sum_a \phi_a \,|a)$

> This all looks similar to AQT.
>
> - In AQT, the result α is possible provided $\langle a\,|\psi\,\rangle \neq 0$.
> - **However**, the Hilbert space inner product in AQT fixes a natural correspondence $|a\rangle \leftrightarrow \langle a|$.

NB: C… he *entire* basis.

- A measurem… This always corre…
- Each measu… al vector (*a* | -- the "effect functional".
- Possibility rule: *a* is possible iff $(a\,|\phi\,) \neq 0$

  This depends only on the state and the effect functional!
- Vectors $|\phi)$ and $c\,|\phi)$ are equivalent ("same state").

Entanglement

A mixed state is a subspace of V.

Two mobits in Z_2-MQT:

$V \otimes V$ contains 16 vectors (15 states), including

- 9 product states $|\alpha, \beta) = |\alpha) \otimes |\beta)$
- 6 entangled states -- e.g., $|R) = |0,0) + |1,1)$

For larger $|F|$ and dim V, entangled states greatly outnumber product states. Most states are entangled.

Hardy's theorem (MQT style): The pattern of possible joint measurement results for a two-mobit entangled state are inconsistent with any theory of local hidden variables.

Advertisement: For details on this and other results, see Mike Westmoreland's poster tomorrow!

Mixed states, etc.

Mixture = collection of possible states: $M = \{|a), |b), \ldots\}$

Mixtures M and M' are equivalent iff they span the same subspace. A mixed state $\langle M \rangle$ is a subspace of V.

Given an entangled state $\left|\Phi^{(12)}\right) = \sum_a \left|a^{(1)}\right) \otimes \left|\phi_a^{(2)}\right)$

then system 2 is in the mixed state $\left\langle \left\{ \left|\phi_a^{(2)}\right) \right\} \right\rangle$

A generalized effect is a subspace $E \subseteq V^*$

E is possible for M iff there exist $(e| \in E$ and $|m) \in M$ such that $(e\,|m\,) \neq 0$.

A generalized measurement is a set of effects $\{E_k\}$ that spans V^*

$$\left\langle \bigcup_k E_k \right\rangle = V^*$$

Bugs and features

<u>What MQT **does not** have</u>

- Probabilities, expectations
- (*F* finite) Continuous sets of states and observables, or continuous time evolution
- Inner product, outer product, orthogonality
- Convexity
- Hermitian conjugation (†)
- Density operators
- Effect operators
- CP maps
- Unextendible product bases

<u>What MQT **does** have</u>

- "Classical" versus "quantum" theories
- Superposition, interference
- Linear dynamics
- Complementary measurements
- Entanglement
- No local hidden variables
- KS theorem, "free will" theorem
- Superdense coding, teleportation, "steering" of mixtures, etc.
- Mixed states, generalized effects, generalized evolution maps
- No cloning theorem
- Nonclassical models of computation

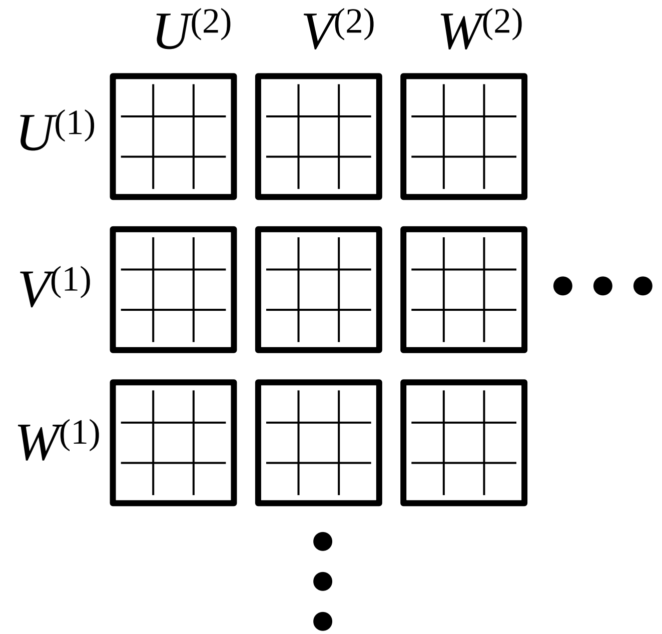
Nothing on the right logically depends on anything on the left.

General models for two systems

General probabilistic models

- Subsystems (1) and (2)
- Measurements *U*, *V*, etc. on each subsystem
- Each joint measurement yields a probability distribution over joint results

$$p\left(u, v | U^{(1)}, V^{(2)}\right)$$



No-signaling principle: The probability of a result on one subsystem does not depend on the choice of measurement on the other subsystem.

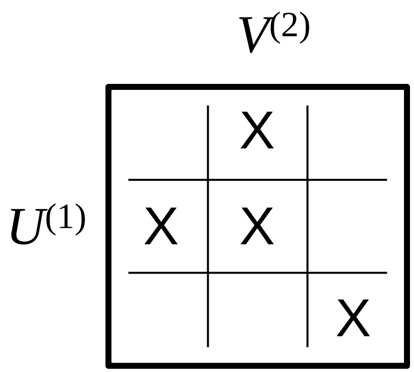
$$\sum_v p\left(u, v | U^{(1)}, V^{(2)}\right) = p\left(u | U^{(1)}\right)$$

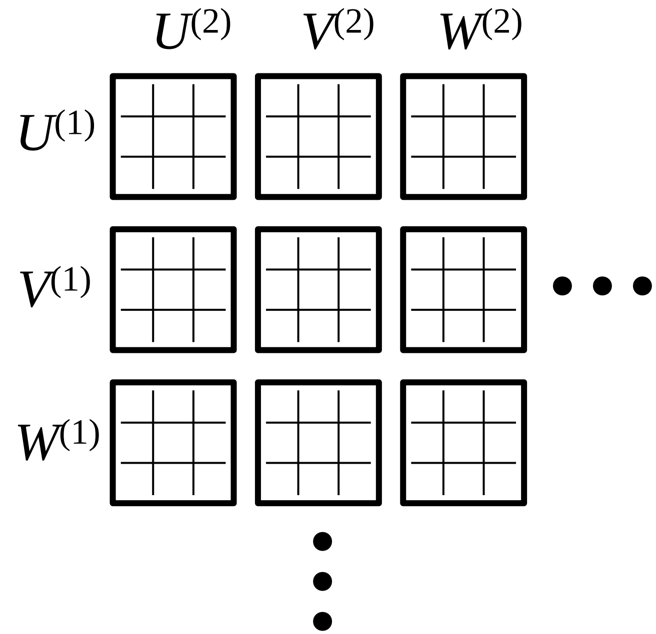
$$\sum_u p\left(u, v | U^{(1)}, V^{(2)}\right) = p\left(v | V^{(2)}\right)$$

Satisfied by AQT

General modal models

- Subsystems (1) and (2)
- Measurements U, V, etc. on each subsystem
- Each joint measurement yields a set of possible joint results



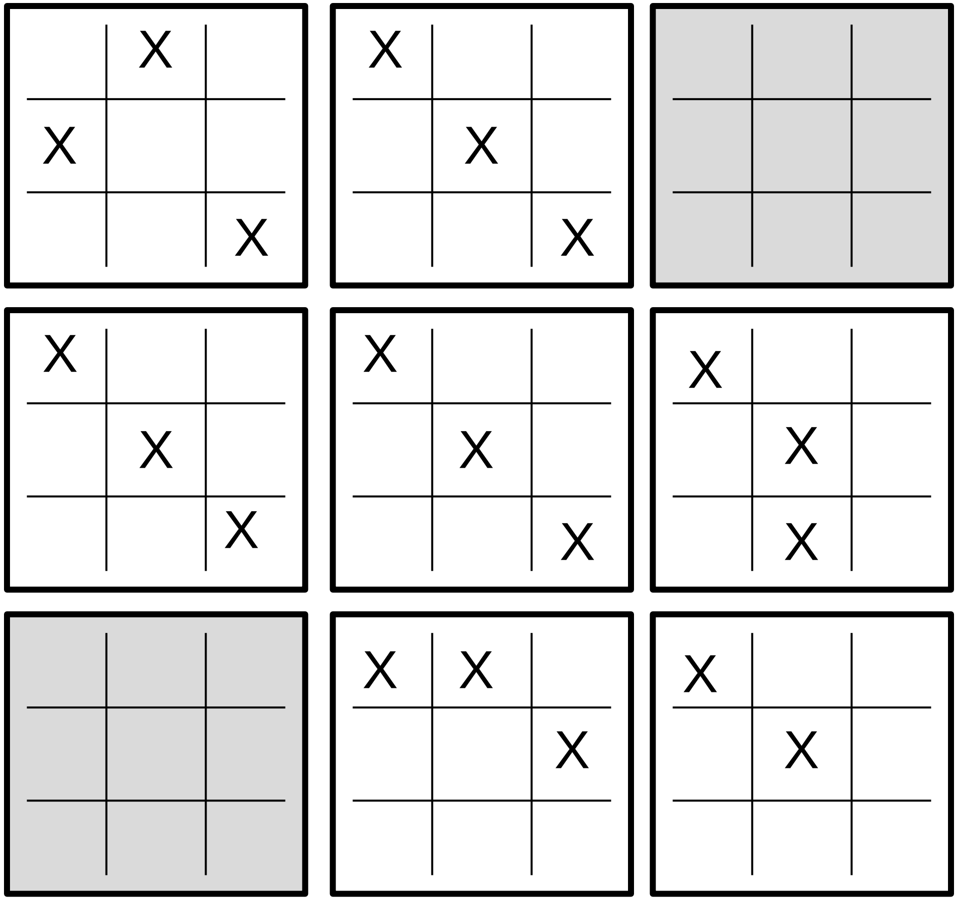


No-signaling principle: The overall possibility of a result on one subsystem does not depend on the choice of measurement on the other subsystem.

Satisfied by MQT

Probabilistic resolution

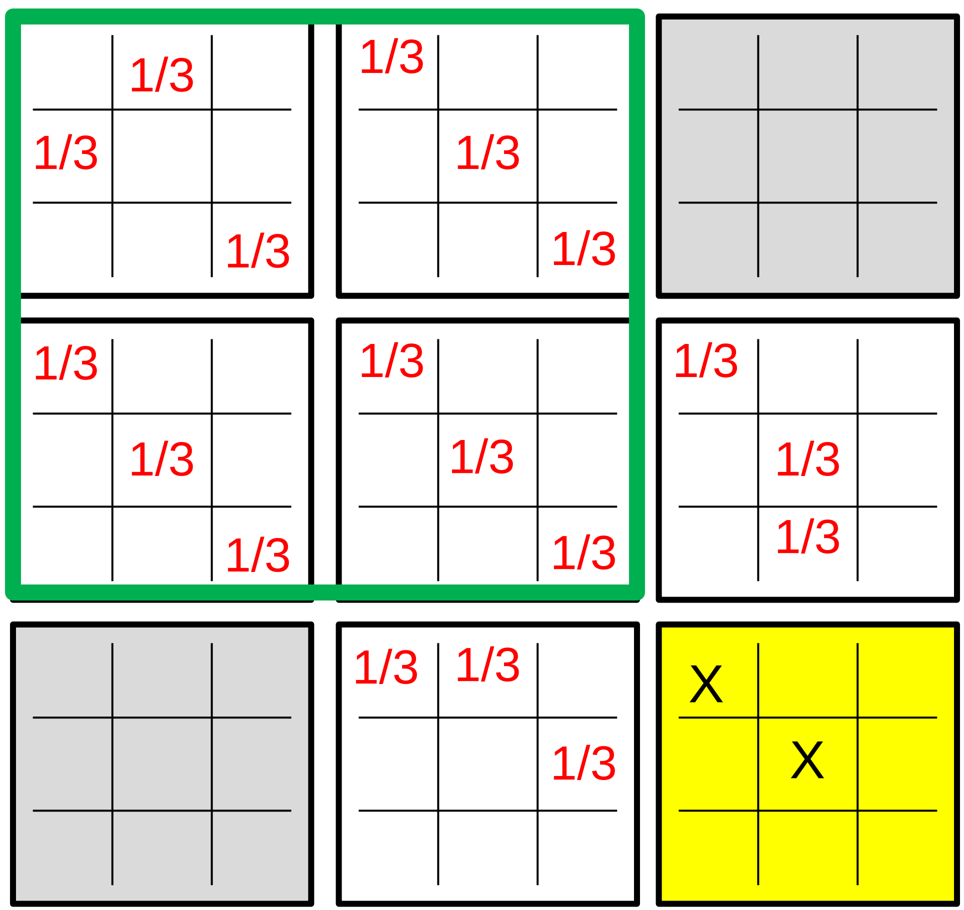
Can we always replace the X's with probabilities that satisfy the no-signaling principle?



From possibility to probability?

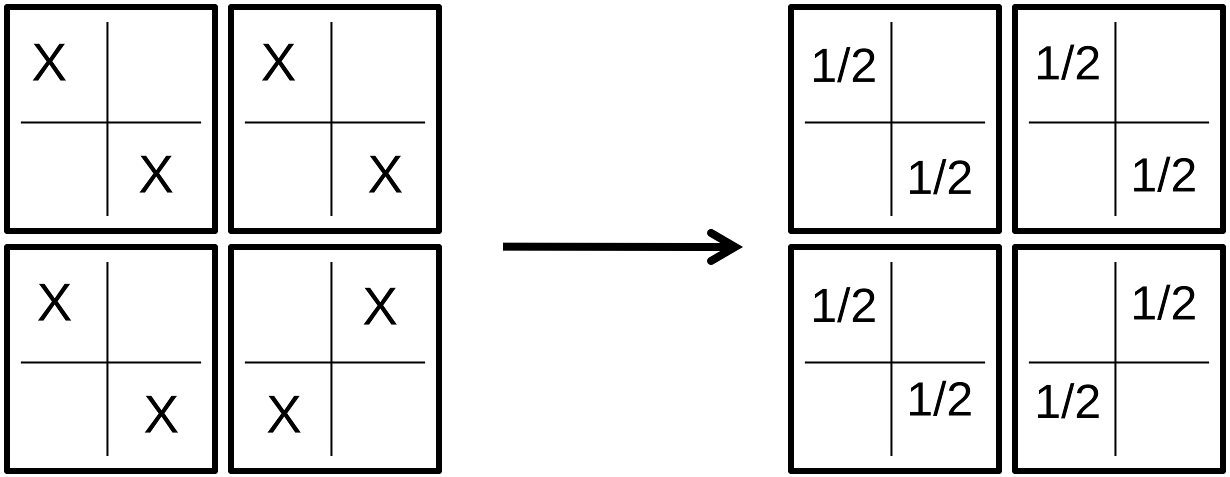
Can we always replace the X's with probabilities that satisfy the no-signaling principle?

All of these probabilities are forced to 1/3.



We cannot match this pattern with any probability assignment.

PR boxes



Popescu-Rohrlich
"nonlocal box"

- This is not an allowed probability pattern for a pair of systems in AQT (Tsirelson bound).
- Is this an allowed possibility pattern for a pair of systems in MQT?

PR boxes in MQT?

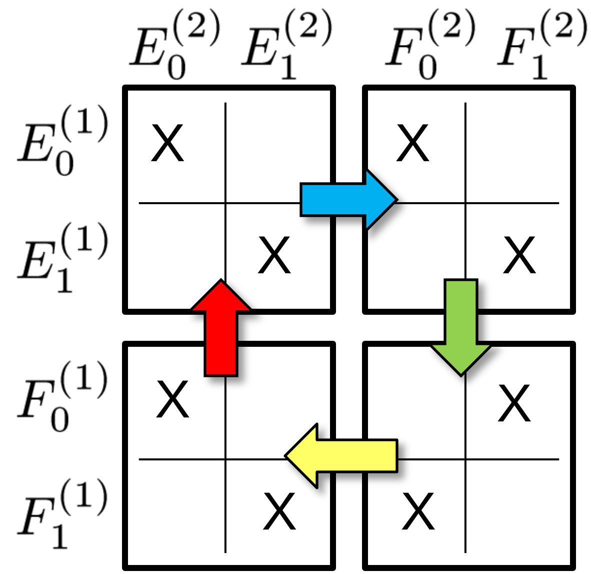
WLOG, we need consider only pure states and "non-overlapping" measurements.

$$E_0^{(1)} \cap E_1^{(1)} = \langle 0 \rangle, \text{ etc.}$$

$$|\Psi_0) \in E_0^{(1)} \otimes E_0^{(2)}$$

$$|\Psi) = |\Psi_0) + |\Psi_1) \neq 0$$

$$|\Psi_1) \in E_1^{(1)} \otimes E_1^{(2)}$$



$$|\Psi_0) \in E_0^{(1)} \otimes F_0^{(2)} \text{ and } |\Psi_1) \in E_1^{(1)} \otimes F_1^{(2)}$$

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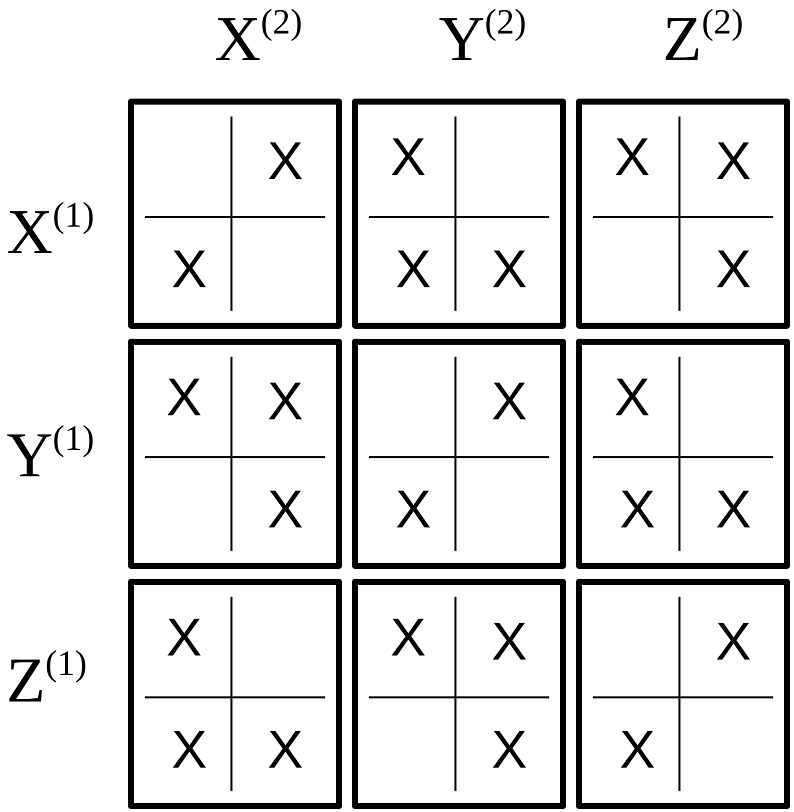
Contradiction!

No PR boxes in MQT!

Two mobits in a "singlet" state $\boxed{|S) = |0,1) - |1,0)}$



- We cannot assign probabilities here such that $p > 0$ for each possible joint result.

Two mobits in a "singlet" state

$$\boxed{|S) = |0,1) - |1,0)}$$

	$X^{(2)}$	$Y^{(2)}$	$Z^{(3)}$
$X^{(1)}$	$\begin{array}{c\|c} & 1/2 \\ \hline 1/2 & \end{array}$	$\begin{array}{c\|c} 1/2 & \\ \hline 0 & 1/2 \end{array}$	$\begin{array}{c\|c} 1/2 & 0 \\ \hline & 1/2 \end{array}$
$Y^{(1)}$	$\begin{array}{c\|c} 1/2 & 0 \\ \hline & 1/2 \end{array}$	$\begin{array}{c\|c} & 1/2 \\ \hline 1/2 & \end{array}$	$\begin{array}{c\|c} 1/2 & \\ \hline 0 & 1/2 \end{array}$
$Z^{(1)}$	$\begin{array}{c\|c} 1/2 & \\ \hline 0 & 1/2 \end{array}$	$\begin{array}{c\|c} 1/2 & 0 \\ \hline & 1/2 \end{array}$	$\begin{array}{c\|c} & 1/2 \\ \hline 1/2 & \end{array}$

- We cannot assign probabilities here such that $p > 0$ for each possible joint result.
- We can assign probabilities if we allow $p = 0$ for some "possible" joint results.

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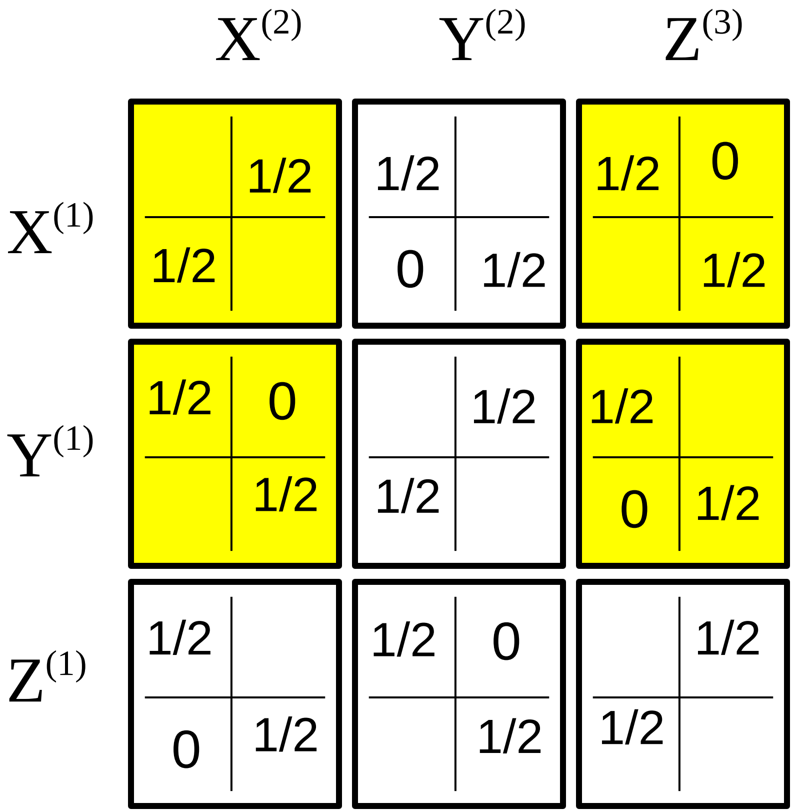

- We cannot assign probabilities here such that $p > 0$ for each possible joint result.
- We can assign probabilities if we allow $p = 0$ for some "possible" joint results.
- The yellow part of the probability table forms a probabilistic PR box!

We cannot simulate this MQT system using AQT.

Types of general modal theory (two systems)

NSP = theories that satisfy the no-signaling principle

SPR = a "strong probabilistic resolution" exists ($p>0$ for each possibility)

WPR = a "weak probabilistic resolution" exists ($p=0$ is okay for some possibilities)

LHV = a local hidden variable theory exists

MQT = possibility pattern can arise in an MQT system

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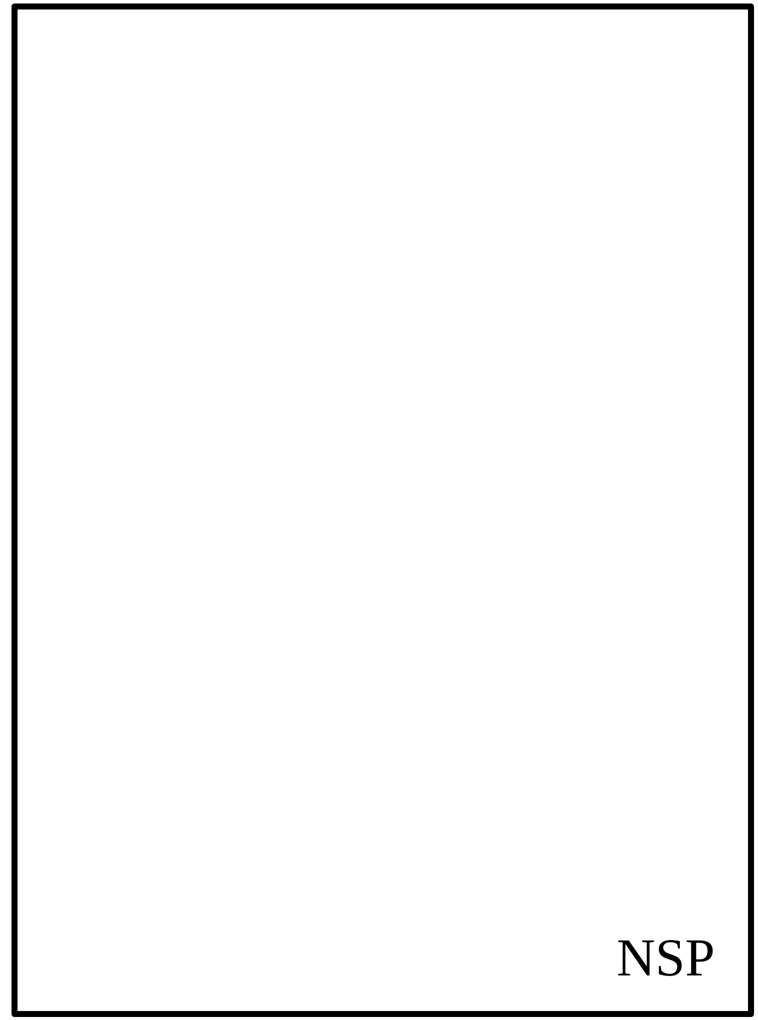

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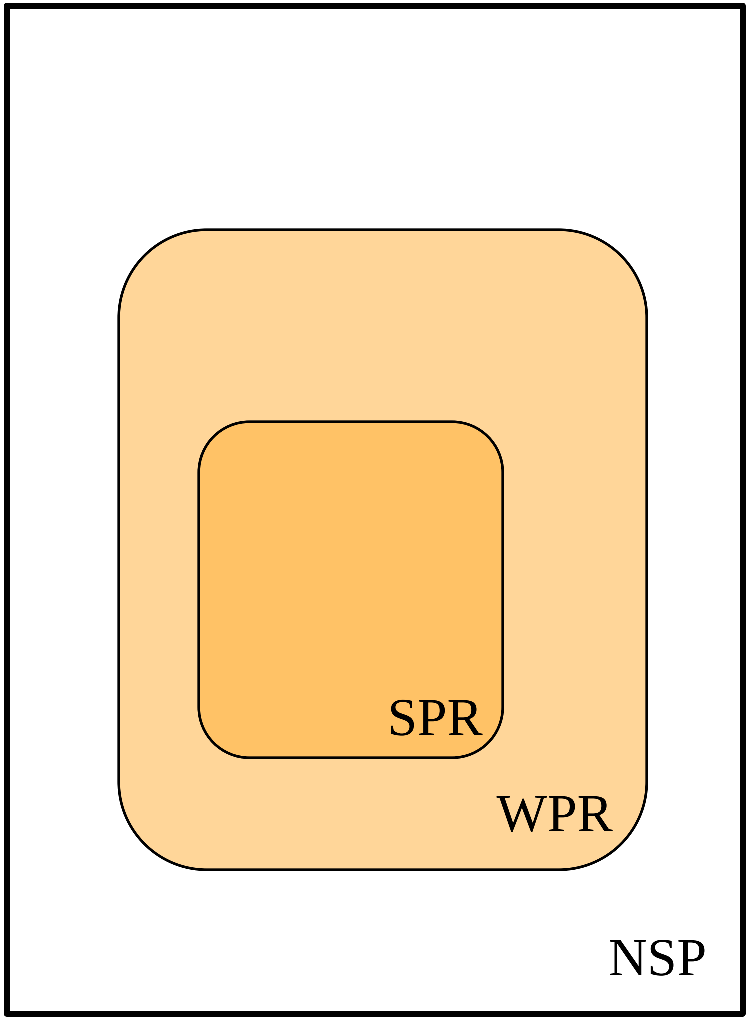

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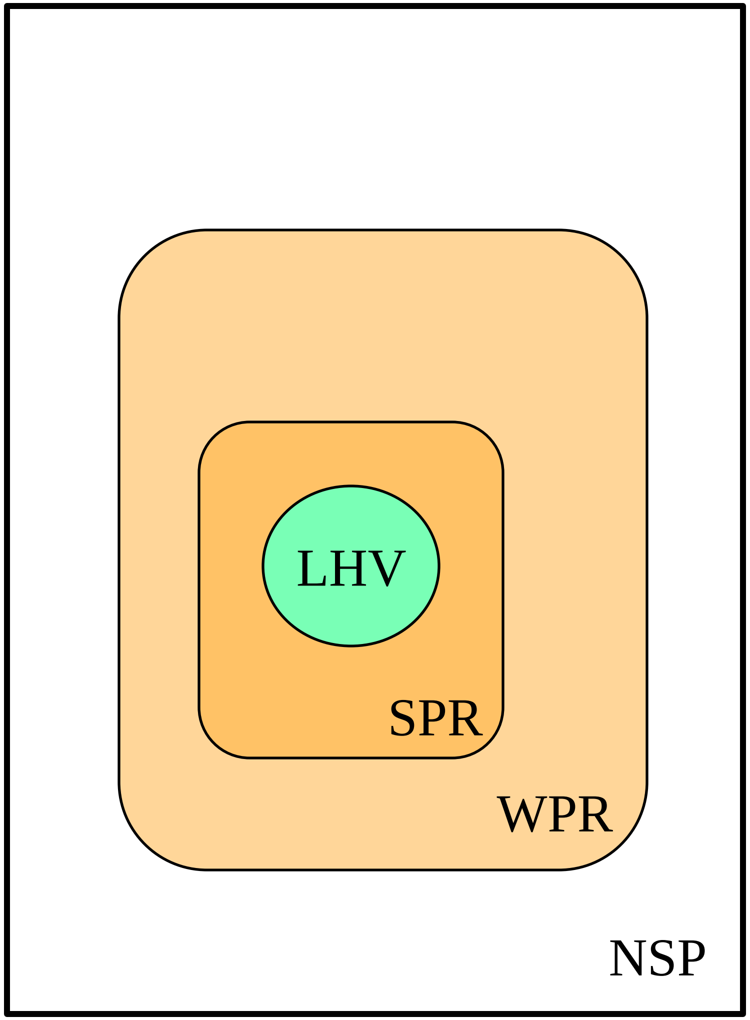


Things we know about MQT

Everything with an LHV model is also in MQT.

Some things in SPR (e.g., PR boxes) are not in MQT.
Some things in MQT (e.g., $|S\rangle$ state) are not in SPR.

Key question: Is everything in MQT also in WPR?

Conjecture: Every two-system MQT model has a weak probabilistic resolution. No WPR counterexamples lie in MQT.

Two simplifications of the problem:

- We only need to consider pure states and effects. (More X's in the tables can only make the WPR problem easier!)
- We only need to consider basic measurements and entangled states with Schmidt number = dim V .

Possibility table for an entangled state

	X		X		X		X	X
X	X		X			X		X
		X		X	X	X	X	
X		X	X	X		X		
	X			X	X		X	X
X		X		X	X		X	
	X		X	X		X	X	
X	X	X	X		X		X	
	X	X		X			X	X

d = dim *V*
N distinct measurements on each system

Our problem: Devise a probability assignment for this table, respecting the NSP.

We may assign *p* = 0 to some of the X's if need be.

What we know: This table corresponds to basic measurements made on a "maximally entangled" MQT state.

Hall's marriage theorem

Two sets of d elements: W = {Alice, Beth, Connie, ... }

M = {Adam, Bob, Carl, ... }

A "compatibility" relation between W and M (subset of $W \times M$)

"Marriage condition": For any n, any subset of n elements of W is compatible with at least n elements of M.

> <u>Theorem</u> (Hall, 1935): If the relation between W and M satisfies the marriage condition, then we can "marry" each element of W with a distinct compatible element of M.

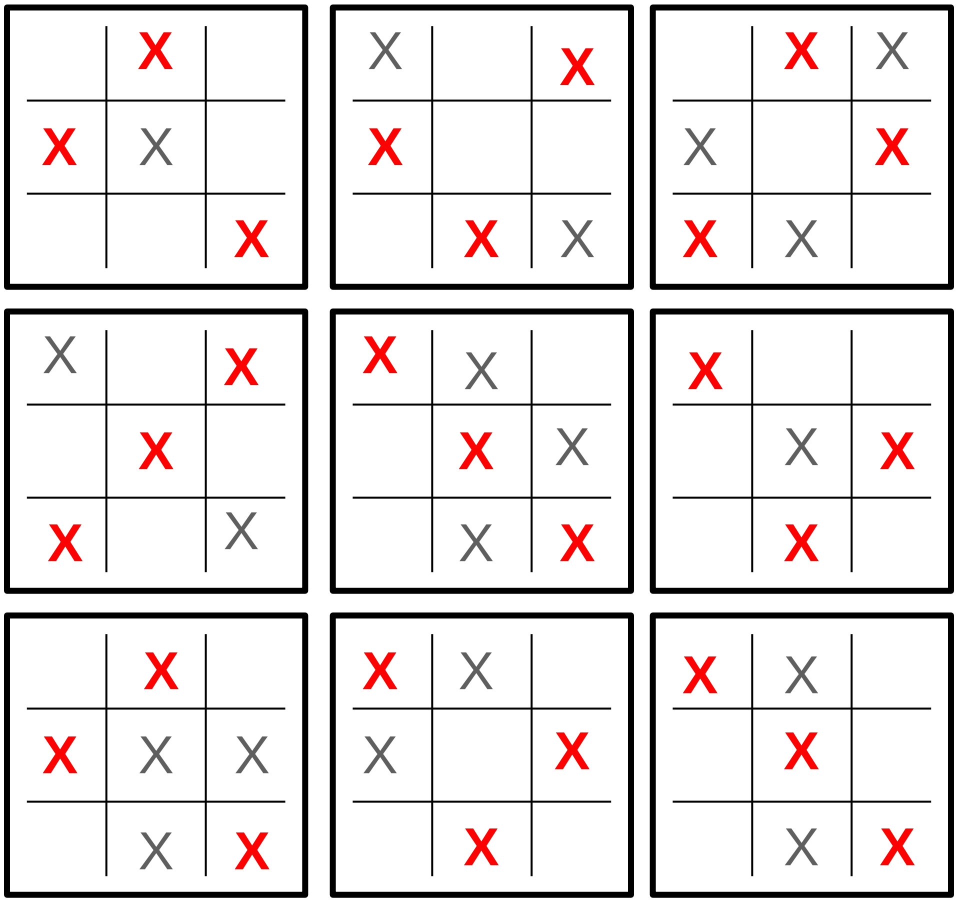
X	X	
	X	X
	X	X

For our entangled MQT state:

Each $d \times d$ sub-table satisfies the marriage condition. Thus, it includes a "permutation" sub-table on d elements.

X	X	
	X	**X**
	X	X

Probability assignment



Do the "marriage trick" on each sub-table in the table.

Assign $p = 1/d$ to each marriage, $p = 0$ to everything else.

In each sub-table, each row and each column sums to $p = 1/d$. Thus, this assignment automatically satisfies the NSP.

Every two-system MQT model has at least one weak probabilistic resolution.

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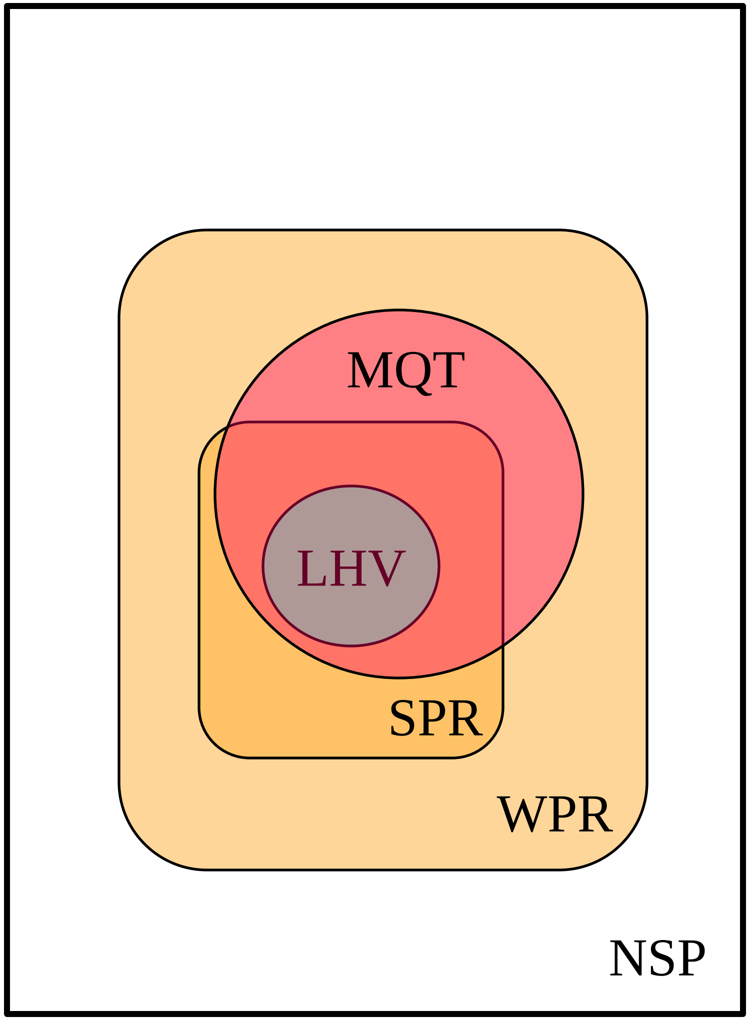

Questions, questions, ...

- We have had to make a strange distinction between "impossible" results and "$p=0$" results. What does this mean?
- What about three or more systems in MQT?
- What quantum information ideas can be adapted to MQT? (Note: Usually, only zero-error problems make sense.)
- Axiom systems presented here so far (Hardy, CDP, etc.) all depend on probabilities:
  - States are defined by probabilities.
  - Convexity of the state space (mixtures of preparations).
  - Effects are linear functionals on states.
- Can these axioms be modified in a sensible way to explore modal theories?
  - If so, does some interesting set of axioms lead to MQT?
  - If so, what kind of MQT (i.e., which scalar field F)?
- MQT is at least weakly consistent with probabilistic theories. Can MQT simulate AQT?

The End

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