Quantum Error Correction / CO639

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lecture: 2004-01-08

1 Which errors can be corrected with the 9-qubit code?

Cor.: The 9-qubit code corrects any 2×2 matrix, or in fact any single-qubit superoperator.

Proof: Let S be a superoperator defined by

$$S: \ \rho \ \mapsto \ \sum_k A_k \rho A_k^\dagger$$

with $\sum_k A_k^\dagger A_k = \mathbb{I}$.

$$\begin{aligned} S \otimes \mathbb{I}^{\otimes 8} (|\overline{\psi}\rangle\langle\overline{\psi}|) \quad &= \quad \textstyle\sum_k A_k |\overline{\psi}\rangle\langle\overline{\psi}| A_k^\dagger \\ &\overset{EC}{\mapsto} \quad |\overline{\psi}\rangle\langle\overline{\psi}| \end{aligned}$$

where $S \otimes \mathbb{I}^{\otimes 8} (|\overline{\psi}\rangle\langle\overline{\psi}|)$ is a mixture of $\{A_k|\overline{\psi}\rangle\}$ with probabilities $\langle\overline{\psi}|A_k^\dagger A_k|\overline{\psi}\rangle$, and EC performs the procedure $A_k|\overline{\psi}\rangle \ \mapsto \ |\overline{\psi}\rangle$. This follows from the Cor. from last lecture.

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2 Error probability with and without decoding

2.1 Classical errors occur

Each qubit is disturbed:

Prob. $\frac{p}{3}$	X error	}	= Prob. p for any error
$\frac{p}{3}$	Y error	}	
$\frac{p}{3}$	Z error	}	
$1-p$	no error		
$(1-p)^9$	no error	}	correctable errors
$9p(1-p)^8$	1 error	}	
$36p^2(1-p)^7$	2 errors	}	may be uncorrectable
	⋮	}	

Total prob. of uncorrectable errors $= O(p^2)$

2.2 Every qubit is a little disturbed

Let $U^{\otimes 9}$ be the error with $U = \mathbb{I} + \epsilon U'$, then we get

$$U^{\otimes 9} = (\mathbb{I}^{\otimes 9} + \epsilon \underbrace{(U' \otimes \mathbb{I}^{\otimes 8} + \mathbb{I} \otimes U' \otimes \mathbb{I}^{\otimes 7} + \cdots)}_{\text{correctable}} + \epsilon^2 \underbrace{(U' \otimes U' \otimes \mathbb{I}^{\otimes 7}) + \cdots}_{\text{uncorrectable}}$$

So the final state is of the form $(\cdots)|\overline{\psi}\rangle + O(\epsilon^2)|\,?\rangle$

If ϵ is small, this gives a high fidelity to the original state: $\sim 1 - O(\epsilon^2)$.

3 Necessary and sufficient conditions for error correction

For error correction, we have the two following steps:

1. Identify error

2. Correct / invert error

For these steps, the following conditions are sufficient:

1. $E|\overline{\psi}\rangle,\ F|\overline{\psi}\rangle$ orthogonal or the same:

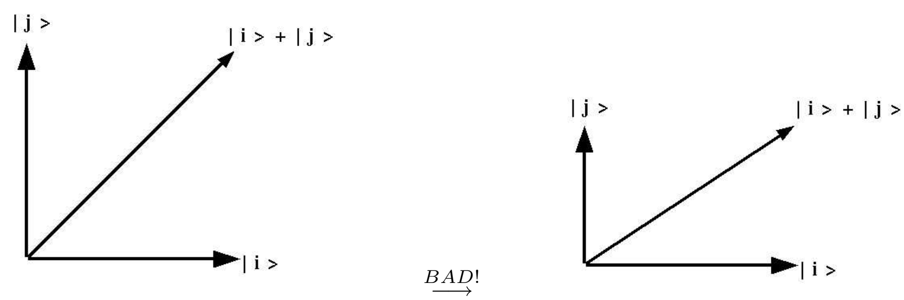
   (a) $\langle\overline{\psi}\,|E^\dagger F|\overline{\psi}\rangle = 0$ or

   (b) $E|\overline{\psi}\rangle = F|\overline{\psi}\rangle\ \forall\overline{\psi}$
   $\Leftrightarrow\ (E - F)|\overline{\psi}\rangle = 0$

2. $|\overline{\psi}\rangle \mapsto E|\overline{\psi}\rangle$
   Mapping must be "unitary" restricted to valid encoded states. Take an orthogonal basis $|\bar{i}\rangle$

   (a) $E|\bar{i}\rangle \perp E|\bar{j}\rangle$, if $i \neq j$.
   $\Leftrightarrow\ \langle\bar{i}\,|E^\dagger E|\bar{j}\rangle = 0$

   (b)



$||E|\,\bar{i}\rangle|| = ||E|\,\bar{j}\rangle||$

$\Leftrightarrow\ \langle\bar{i}\,|E^\dagger E|\,\bar{i}\rangle = \langle\bar{j}\,|E^\dagger E|\,\bar{j}\rangle$

Otherwise angles change, as in the figure, and the operation is non-unitary.

Theorem: QECC $|\,\psi\rangle \mapsto |\,\overline{\psi}\rangle$ corrects $\mathcal{E}$ spanned by $\{E_a\}$ if and only if there exists (c_{ab}) s. t.

$$\langle\bar{j}\,|E_a^\dagger E_b|\,\bar{i}\rangle = c_{ab}\delta_{ij}.$$

Proof:

- Sufficiency:

  2. ok

  1. c_{ab} is Hermitian and so diagonalizable.

  Define new coordinates by

$$\sum_a \alpha_{ca} E_a = F_c \ \text{ s. t.}$$
$$\langle\bar{j}\,|F_c^\dagger F_d|\,\bar{i}\rangle = \tilde{c}_{cd}\delta_{ij}$$

  Set $\tilde{c}_{cd} = \delta_{cd}\ \tilde{c}_c$. Then 1. is ok.

- Necessary condition: 2. must always hold.

  – $\langle\bar{j}\,|E_a^\dagger E_b|\,\bar{i}\rangle = 0$ if $i \neq j$:

  Pf: If not, $\exists\ a$, b, i, j s. t.

$$E_a|\,\bar{i}\rangle \not\perp E_b|\,\bar{j}\rangle$$

  Then there exists no EC mapping

$$E_a|\,\bar{i}> \ \mapsto\ |\,\bar{i}>$$
$$E_b|\,\bar{j}> \ \mapsto\ |\,\bar{j}>$$

  – $\langle\bar{i}\,|E_a^\dagger E_b|\,\bar{i}\rangle = c_{ab}$

  Pf: Suppose not, then:

  $\exists\ i$, j s. t.

$$Re\langle\bar{i}\,|E_a^\dagger E_b|\,\bar{i}\rangle \neq Re\langle\bar{j}\,|E_a^\dagger E_b|\,\bar{j}\rangle$$

$$\begin{aligned}\langle\bar{i}\,|(E_a^\dagger + E_b^\dagger)(E_a + E_b)|\,\bar{i}\rangle \\ &= \langle\bar{i}\,|E_a^\dagger E_a|\,\bar{i}\rangle + \langle\bar{i}\,|E_b^\dagger E_b|\,\bar{i}\rangle + 2Re\langle\bar{i}\,|E_a^\dagger E_b|\,\bar{i}\rangle \\ &= \langle\bar{j}\,|E_a^\dagger E_a|\,\bar{j}\rangle + \langle\bar{j}\,|E_b^\dagger E_b|\,\bar{j}\rangle + 2Re\langle\bar{j}\,|E_a^\dagger E_b|\,\bar{j}\rangle\end{aligned}$$

$$\Longrightarrow Re\langle\bar{i}\,|E_a^\dagger E_b|\,\bar{i}\rangle = Re\langle\bar{j}\,|E_a^\dagger E_b|\,\bar{j}\rangle$$

which contradicts the assumption. (We have used 2b above.)

Similar for imaginary part.

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Def.: If $\langle\bar{i}\,|F_c^\dagger F_c|\,\bar{i}\rangle = 0$, then 0 is an eigenvalue for c_{ab} and the QECC is called **degenerate**. If 0 is not an eigenvalue for c_{ab}, then the QECC is called **nondegenerate**.

Def.: The **weight** $wt(E)$ of an error E is the number of qubits where E is not the identity.

Remark: Consider tensor products of $\mathbb{I}$, X, Y, Z of $wt \leq t$. Then

$$\{E_a^\dagger E_b\} = \text{tensor products of } \mathbb{I},\, X,\, Y,\, Z \text{ of } wt \leq 2t$$

or

$$\langle\bar{j}\,|P|\,\bar{i}\rangle = c(P)\delta_{ij} \text{ with } wt(P) \leq 2t$$

Def: The **distance** of a QECC is the minimum weight of P s. t. $\langle\bar{j}\,|P|\,\bar{i}\rangle \neq c(P)\,\delta_{ij}$.

Remark: A code of distance $2t+1$ corrects t errors.