Problem Set #1

CO 639: Quantum Error Correction
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Due Tues., Jan. 27

Problem 1. Uncorrectable Errors and the Nine-Qubit Code

While the nine-qubit code can correct an arbitrary single-qubit error, it can also correct some multiple-qubit errors. In what follows, use

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

and X_i, Y_i, or Z_i represents X, Y, or Z applied to the ith qubit.

a) Which of the following errors can be corrected by the nine-qubit code: X_1X_3, X_2X_7, X_5Z_6, Z_5Z_6, Y_2Z_8?

b) Suppose we perform the usual error correction procedure on the nine-qubit code after one of the above two-qubit errors has occurred. This returns us to an encoded state, but it may not be the correct encoded state. For those errors that cannot be corrected, calculate the operation that is performed on the encoded state. That is, if we start with $\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle$, what state do we end up with?

Problem 2. Maximally Entangled States and QECCs

In this problem, we will see a relationship between a certain class of highly entangled states and quantum error-correcting codes with a large distance. In particular, we can define a "QECC" that encodes 0 qubits as a state which satisfies a restricted version of the QECC conditions: An n-qubit state $|\psi\rangle$ is an $[[n, 0, d]]$ code if it satisfies

$$\langle\psi|E|\psi\rangle = \delta_{E,I} \tag{2}$$

for all Pauli operators E with $\mathrm{wt}(E) < d$.

a) We define a maximally entangled state as an n-qubit state $|\psi\rangle$ with the property that for $r \leq n/2$, the density matrix of any r qubits is the identity. That is, any subset of r qubits is maximally entangled with the remaining $n-r$ qubits. Show that the GHZ state $|000\rangle + |111\rangle$ is maximally entangled. Write down the stabilizer for the GHZ state.

b) Show that a maximally entangled state is an $[[n, 0, \lfloor n/2 \rfloor + 1]]$ QECC.

c) Show that any $[[n, 0, \lfloor n/2 \rfloor + 1]]$ code is a maximally entangled state.

d) Find a 5-qubit maximally entangled state.

e) Why did I need to make a special definition for an $[[n, 0, d]]$ code? That is, why couldn't I just have taken the $k = 0$ case of the QECC conditions for a general $[[n, k, d]]$ code?

Problem 3. Other Forms and Consequences of the QECC Conditions

a) Show that the QECC conditions
$$\langle\bar{j}|E_a^\dagger E_b^\dagger|\bar{i}\rangle = C_{ab}\delta_{ij} \tag{3}$$
are equivalent to
$$\langle\overline{\psi}|E_a^\dagger E_b^\dagger|\overline{\psi}\rangle = C_{ab}, \tag{4}$$
where $|\overline{\psi}\rangle$ is any encoded state, not just a basis state. What if we only required $|\overline{\psi}\rangle$ to run over basis states — would that also be equivalent?

b) Suppose that a QECC has distance d. Show that the density matrix for any $d-1$ qubits of the code is independent of the encoded state.

c) Suppose we have a linear encoding $|\psi\rangle \mapsto |\overline{\psi}\rangle$, mapping k qubits into n qubits, and suppose that the encoding has the property that for any $d-1$ qubits of the code, the density matrix is independent of the encoded state. Show that the encoding forms a QECC with distance at least d.

Problem 4. Correcting X and Z, but not Y

a) Find a stabilizer code encoding 3 qubits in 8 qubits which corrects any X or Z error on a single qubit, but does not correct a single-qubit Y error on any qubit. What is the distance of this code?

b) Suppose you have another QECC with the property that it corrects any single X or Z error on a single qubit, but fails to correct some single-qubit Y error. What can you say about its distance?

Problem 5. Combining Stabilizer Codes

Suppose we have two stabilizer codes S_1 (an $[[n_1, k_1, d_1]]$ code) and S_2 (an $[[n_2, k_2, d_2]]$ code). Suppose S_1 has generators $M_1, \ldots, M_{n_1-k_1}$ and S_2 has generators $N_1, \ldots, N_{n_2-k_2}$.

a) Suppose we take the generators $M_i \otimes I_{n_2}$ and $I_{n_1} \otimes N_j$ (where I_n is the identity acting on n qubits). Show that these generators define an Abelian group and therefore give an $(n_1 + n_2)$-qubit stabilizer code. Calculate k, the number of encoded qubits in the code. What can you say about its distance?

b) Now suppose $n_1 - k_1 = n_2 - k_2$, and we instead take generators $M_i \otimes N_i$. Do they define a QECC? If so, what are k and d for this code?

c) Suppose $n_1 = n_2 = n$, and we take generators M_i *and* N_j, for a total of $2n - (k_1 + k_2)$ generators. Do they define a QECC? If so, what are k and d for this code?

Problem 6. Ket Representation of Stabilizer Codes

a) Write out two basis states $|\overline{0}\rangle$ and $|\overline{1}\rangle$ for the 5-qubit code using the usual ket notation.

b) Find the stabilizer for the subspace spanned by the following states:
$$|\overline{0}\rangle = |000\rangle + i|110\rangle + i|011\rangle - |101\rangle \tag{5}$$
$$|\overline{1}\rangle = |111\rangle + i|001\rangle + i|100\rangle - |010\rangle. \tag{6}$$