Assignment: Quantum strategies

Due at 23:59 on Friday, May 25, 2012.

There are 30 marks available in this assignment. The assignment will be graded out of 20.

1. **Co-strategies are adjoint channels.** *[5 marks.]* Let $\Phi : \mathrm{L}(\mathcal{X}) \to \mathrm{L}(\mathcal{Y})$ be any super-operator. By analogy with matrices, the *adoint* of Φ is the unique super-operator $\Phi^* : \mathrm{L}(\mathcal{Y}) \to \mathrm{L}(\mathcal{X})$ satisfying $\langle \Phi(X), Y \rangle = \langle X, \Phi^*(Y) \rangle$ for all $X \in \mathrm{L}(\mathcal{X})$ and $Y \in \mathrm{L}(\mathcal{Y})$.

   It is easy to see that if $\Phi : X \mapsto \sum_i A_i X A_i^*$ then $\Phi^* : Y \mapsto \sum_i A_i^* Y A_i$ and you may use this fact without proof in your answer.

   Prove that if Φ is completely positive then $J(\Phi^*) = W J(\Phi)^{\mathsf{T}} W^*$ where

   $$W : \mathcal{Y} \otimes \mathcal{X} \to \mathcal{X} \otimes \mathcal{Y} : |j\rangle|i\rangle \mapsto |i\rangle|j\rangle$$

   is the unitary swap operator.

   Remark. Let $J(\Xi)$ be an $(r+1)$-round strategy for input spaces $\mathbb{C}, \mathcal{Y}_1, \dots, \mathcal{Y}_r$ and output spaces $\mathcal{X}_1, \dots, \mathcal{X}_r, \mathbb{C}$ where the channel $\Xi : \mathrm{L}(\mathcal{Y}_{1\dots r}) \to \mathrm{L}(\mathcal{X}_{1\dots r})$ is of the form described in lecture.

   In lecture we defined an r-round co-strategy for input spaces $\mathcal{X}_1, \dots, \mathcal{X}_r$ and output spaces $\mathcal{Y}_1, \dots, \mathcal{Y}_r$ by $W J(\Xi)^{\mathsf{T}} W^*$, which you now know is equal to $J(\Xi^*)$. In other words, the Choi-Jamiołkowski representation of a co-strategy is simply the Choi matrix of the *adjoint* Ξ^* of the super-operator Ξ described in lecture.

   Solution. We claim that $W \operatorname{vec}(A) = \operatorname{vec}(A^{\mathsf{T}})$ for any operator $A : \mathcal{X} \to \mathcal{Y}$. To see this, write A as a linear combination of standard basis matrices:

   $$A = \sum_{i=1}^{\dim(\mathcal{X})} \sum_{j=1}^{\dim(\mathcal{Y})} \alpha_{i,j} |j\rangle\langle i|$$

   so that

   $$\operatorname{vec}(A) = \sum_{i,j} \alpha_{i,j} |j\rangle|i\rangle.$$

   Then

   $$W \operatorname{vec}(A) = \sum_{i',j'} \sum_{i,j} \alpha_{i,j} |i'\rangle|j'\rangle \underbrace{\langle j'|\langle i'||j\rangle|i\rangle}_{\delta_{(i,j),(i',j')}} = \sum_{i,j} \alpha_{i,j} |i\rangle|j\rangle = \operatorname{vec}(A^{\mathsf{T}})$$

   as claimed.

   We all know that $(|\psi\rangle\langle\psi|)^{\mathsf{T}} = |\overline{\psi}\rangle\langle\overline{\psi}|$ and so it also holds that $(\operatorname{vec}(A)\operatorname{vec}(A)^*)^{\mathsf{T}} = \operatorname{vec}(\overline{A})\operatorname{vec}(\overline{A})^*$.

   Since Φ is completely positive we may write

   $$\Phi : X \mapsto \sum_{i=1}^{n} A_i X A_i^*$$

for some choice of $A_1, \ldots, A_n : \mathcal{X} \to \mathcal{Y}$. We know from lecture that

$$J(\Phi) = \sum_{i=1}^{n} \mathrm{vec}(A_i)\,\mathrm{vec}(A_i)^*$$

and so

$$J(\Phi)^{\mathsf{T}} = \sum_{i=1}^{n} \mathrm{vec}(\overline{A_i})\,\mathrm{vec}(\overline{A_i})^*.$$

Then

$$WJ(\Phi)^{\mathsf{T}}W^* = \sum_{i=1}^{n} \mathrm{vec}(\overline{A_i}^{\mathsf{T}})\,\mathrm{vec}(\overline{A_i}^{\mathsf{T}})^* = \sum_{i=1}^{n} \mathrm{vec}(A_i^*)\,\mathrm{vec}(A_i^*)^*.$$

The result follows from the fact that the adjoint Φ^* is given by $\Phi^* : Y \mapsto \sum_{i=1}^{n} A_i^* Y A_i$.

2. **Characterization of measuring strategies.** *[5 marks.]* Prove that if $\{Q_a\} \subset \mathrm{Pos}(\mathcal{Y}_{1\ldots r} \otimes \mathcal{X}_{1\ldots r})$ is a finite set for which $\sum_a Q_a$ is an r-round non-measuring strategy for input spaces $\mathcal{X}_1, \ldots, \mathcal{X}_r$ and output spaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_r$ then $\{Q_a\}$ is an r-round measuring strategy for the same input and output spaces.

   Solution. Let $\mathcal{M}$ be a space with dimension equal to the number of elements of $\{Q_a\}$ so that $\{|a\rangle\} \subset \mathcal{M}$ is an orthonormal basis for $\mathcal{M}$ and $\{|a\rangle\langle a|\} \subset \mathrm{Meas}(\mathcal{M})$ is a measurement. Define $Q \in \mathrm{Pos}(\mathcal{M} \otimes \mathcal{Y}_{1\ldots r} \otimes \mathcal{X}_{1\ldots r})$ by

$$Q = \sum_a |a\rangle\langle a| \otimes Q_a. \tag{1}$$

   We claim that Q is an r-round non-measuring strategy for input spaces $\mathcal{X}_1, \ldots, \mathcal{X}_r$ and output spaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_{r-1}, \mathcal{Y}_r \otimes \mathcal{M}$. In particular, we have

$$\mathrm{Tr}_{\mathcal{Y}_r \otimes \mathcal{M}} \left(\sum_a |a\rangle\langle a| \otimes Q_a \right) = \mathrm{Tr}_{\mathcal{Y}_r} \left(\sum_a Q_a \right). \tag{2}$$

   Since $\sum_a Q_a$ is a non-measuring strategy for input spaces $\mathcal{X}_1, \ldots, \mathcal{X}_r$ and output spaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_r$, the right side equals $I_{\mathcal{X}_r} \otimes R$ for some $(r-1)$-round non-measuring strategy R for input spaces $\mathcal{X}_1, \ldots, \mathcal{X}_{r-1}$ and output spaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_{r-1}$. The claim now follows from the characterization of non-measuring strategies.

   Let $\mathcal{Z}_1, \ldots, \mathcal{Z}_{r-1}$ be memory spaces and let $\Phi_1, \ldots, \Phi_r$ be channels of the form

$$\Phi_1 : \mathrm{L}(\mathcal{X}_1) \to \mathrm{L}(\mathcal{Y}_1 \otimes \mathcal{Z}_1) \tag{3}$$
$$\Phi_i : \mathrm{L}(\mathcal{X}_i \otimes \mathcal{Z}_{i-1}) \to \mathrm{L}(\mathcal{Y}_i \otimes \mathcal{Z}_i) \qquad (2 \leq i \leq r-1) \tag{4}$$
$$\Phi_r : \mathrm{L}(\mathcal{X}_r \otimes \mathcal{Z}_{r-1}) \to \mathrm{L}(\mathcal{Y}_r \otimes \mathcal{M}) \tag{5}$$

   for which $Q = J(\Phi_r \circ \cdots \circ \Phi_1)$. For each outcome a define

$$\Gamma_a : X \mapsto \mathrm{Tr}_{\mathcal{M}}\left((|a\rangle\langle a| \otimes I_{\mathcal{Y}_{1\ldots r}})X\right) \tag{6}$$

   and note that

$$Q_a = \Gamma_a(Q) = \Gamma_a(J(\Phi_r \circ \cdots \circ \Phi_1)) = J\left(\Gamma_a \circ \Phi_r \circ \cdots \circ \Phi_1\right). \tag{7}$$

   By definition, $\{Q_a\} = \{J(\Gamma_a \circ \Phi_r \circ \cdots \circ \Phi_1)\}$ is an r-round measuring strategy for input spaces $\mathcal{X}_1, \ldots, \mathcal{X}_r$ and output spaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_r$ (with final memory space $\mathcal{M}$).

3. **Lower-bound for quantum coin-flipping.** *[5 marks.]* Consider a scenario in which Alice and Bob wish to flip a fair coin but do not trust each other. A *quantum coin-flipping protocol with cheating probability* p is a specification of an r-round measuring strategy $\{A_0, A_1\}$ for Alice and an r-round measuring co-strategy $\{B_0, B_1\}$ for Bob with the following properties:

   Honest parties flip fair coins.
   $\langle A_0, B_0\rangle = \langle A_1, B_1\rangle = \frac{1}{2}$.

   Cheaters can't cheat too much.
   For any outcome $i \in \{0, 1\}$ we have the following:
   For every r-round measuring co-strategy $\{B'_0, B'_1\}$ for Bob it holds that $\langle A_i, B'_i\rangle \leq p$.
   For every r-round measuring strategy $\{A'_0, A'_1\}$ for Alice it holds that $\langle A'_i, B_i\rangle \leq p$.

   By definition, every protocol must have $p \geq 1/2$. Use the formula for maximum output probabilities to prove that every quantum coin-flipping protocol has cheating probability $p \geq 1/\sqrt{2} \approx 0.707$.

   Hint. Fix $i \in \{0, 1\}$ and let p be the maximum probability that a cheating Bob can force honest-Alice to output i. By the formula for maximum output probabilities we know there exists a strategy Q for Alice with $A_i \preceq pQ$. What happens if a cheating Alice uses strategy Q against honest-Bob?

   Solution. Fix $i \in \{0, 1\}$ and let p be the maximum probability that a cheating Bob can force honest-Alice to output i. Obviously we have $p \geq 1/2$. By the formula for maximum output probabilities we know there exists a strategy Q for Alice with $A_i \preceq pQ$. If a cheating Alice plays this strategy Q then honest-Bob outputs i with probability

   $$\langle Q, B_i\rangle \geq \frac{1}{p}\langle A_i, B_i\rangle = \frac{1}{2p}.$$

   Given that $\max\{p, 1/2p\} \geq 1/\sqrt{2}$ for all $p \geq 0$ we have that either honest-Alice or honest-Bob can be convinced to output i with probability at least $1/\sqrt{2}$.

4. **Channel-channels are two-round strategies.** We all know that channels map quantum states to quantum states and that a super-operator Φ is a channel if and only if Φ is completely positive and trace-preserving. But what kind of mappings map quantum *channels* to quantum *channels*?

   Let $\mathrm{T}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}})$ denote the space of super-operators of the form $\Phi : \mathrm{L}(\mathcal{H}_{\text{in}}) \to \mathrm{L}(\mathcal{H}_{\text{out}})$. A mapping

   $$\mathcal{C} : \mathrm{T}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}) \to \mathrm{T}(\mathcal{K}_{\text{in}}, \mathcal{K}_{\text{out}}) \tag{8}$$

   is called *completely completely positive* if for every choice of spaces $\mathcal{X}_{\text{in}}, \mathcal{X}_{\text{out}}$ and every choice of completely positive super-operators $\Psi : \mathrm{L}(\mathcal{H}_{\text{in}} \otimes \mathcal{X}_{\text{in}}) \to \mathrm{L}(\mathcal{H}_{\text{out}} \otimes \mathcal{X}_{\text{out}})$ it holds that the super-operator

   $$\left(\mathcal{C} \otimes \mathbb{1}_{\mathrm{T}(\mathcal{X}_{\text{in}}, \mathcal{X}_{\text{out}})}\right)(\Psi) : \mathrm{L}(\mathcal{K}_{\text{in}} \otimes \mathcal{X}_{\text{in}}) \to \mathrm{L}(\mathcal{K}_{\text{out}} \otimes \mathcal{X}_{\text{out}}) \tag{9}$$

   is also completely positive. (Here $\mathbb{1}_{\mathrm{T}(\mathcal{X}_{\text{in}}, \mathcal{X}_{\text{out}})}$ denotes the identity mapping on $\mathrm{T}(\mathcal{X}_{\text{in}}, \mathcal{X}_{\text{out}})$, as you might expect.)

   We say that $\mathcal{C}$ is *trace-preserving-preserving* if for every choice of trace-preserving super-operators $\Phi : \mathrm{L}(\mathcal{H}_{\text{in}}) \to \mathrm{L}(\mathcal{H}_{\text{out}})$ it holds that $\mathcal{C}(\Phi)$ is also trace-preserving. We say that $\mathcal{C}$ is a *channel-channel* if it is both completely completely positive and trace-preserving-preserving.

   For any mapping $\mathcal{C}$ of the form (8) let us define a super-operator $K_{\mathcal{C}}$ so that

   $$K_{\mathcal{C}} : \mathrm{L}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}) \to \mathrm{L}(\mathcal{K}_{\text{out}} \otimes \mathcal{K}_{\text{in}}) : J(\Phi) \mapsto J(\mathcal{C}(\Phi)) \tag{10}$$

(a) **(Challenge problem.)** *[5 marks.]* Prove that if $\mathcal{C}$ is a channel-channel then $J(K_\mathcal{C})$ is a two-round strategy for input spaces $\mathcal{K}_{\text{in}}, \mathcal{H}_{\text{out}}$ and output spaces $\mathcal{H}_{\text{in}}, \mathcal{K}_{\text{out}}$.

Solution. We begin by establishing several properties of the super-operator $K_\mathcal{C}$.

Claim 1. *If $\mathcal{C}$ is completely completely positive then $K_\mathcal{C}$ is completely positive.*

Proof. Let $\mathcal{X}$ be any space, let $P \in \mathrm{Pos}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}} \otimes \mathcal{X})$ be any positive semidefinite operator. We must show that $(K_\mathcal{C} \otimes \mathbb{1}_\mathcal{X})(X)$ is positive semidefinite. To this end let $\Psi : \mathbb{C} \to \mathrm{L}(\mathcal{X})$ be the completely positive super-operator with $J(\Psi) = X$. We have

$$(K_\mathcal{C} \otimes \mathbb{1}_\mathcal{X})(X) = K_{\mathcal{C}\otimes\mathbb{1}_{\mathrm{T}(\mathbb{C},\mathcal{X})}}(J(\Psi)) = J\left(\left(\mathcal{C} \otimes \mathbb{1}_{\mathrm{T}(\mathbb{C},\mathcal{X})}\right)(\Psi)\right). \tag{11}$$

Since $\mathcal{C}$ is completely completely positive, the right side of (11) is positive semidefinite, as desired. □

Claim 2. *If $\mathcal{C}$ is trace-preserving-preserving then $K_\mathcal{C}$ satisfies*

$$\mathrm{Tr}_{\mathcal{H}_{\text{out}}}(X) = I_{\mathcal{H}_{\text{in}}} \implies \mathrm{Tr}_{\mathcal{K}_{\text{out}}}(K_\mathcal{C}(X)) = I_{\mathcal{K}_{\text{in}}} \tag{12}$$

for all $X \in \mathrm{L}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$.

Proof. Let $X \in \mathrm{L}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$ be any operator with $\mathrm{Tr}_{\mathcal{H}_{\text{out}}}(X) = I_{\mathcal{H}_{\text{in}}}$. We must show that $\mathrm{Tr}_{\mathcal{K}_{\text{out}}}(K_\mathcal{C}(X)) = I_{\mathcal{K}_{\text{in}}}$. To this end let $\Phi : \mathrm{L}(\mathcal{H}_{\text{in}}) \to \mathrm{L}(\mathcal{H}_{\text{out}})$ be the trace-preserving super-operator with $J(\Phi) = X$. By definition we have

$$K_\mathcal{C}(X) = K_\mathcal{C}(J(\Phi)) = J(\mathcal{C}(\Phi)). \tag{13}$$

Since $\mathcal{C}$ is trace-preserving-preserving, the partial trace $\mathrm{Tr}_{\mathcal{K}_{\text{out}}}$ of the right side must be $I_{\mathcal{K}_{\text{in}}}$, as desired. □

Claim 3. *If $\mathcal{C}$ is trace-preserving-preserving then $K_\mathcal{C}$ satisfies*

$$\mathrm{Tr}_{\mathcal{H}_{\text{out}}}(Y) = 0_{\mathcal{H}_{\text{in}}} \implies \mathrm{Tr}_{\mathcal{K}_{\text{out}}}(K_\mathcal{C}(Y)) = 0_{\mathcal{K}_{\text{in}}} \tag{14}$$

for all $Y \in \mathrm{L}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$.

Proof. Let $X \in \mathrm{L}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$ be such that $\mathrm{Tr}_{\mathcal{H}_{\text{out}}}(X) = I_{\mathcal{H}_{\text{in}}}$ and let $Y \in \mathrm{L}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$ be such that $\mathrm{Tr}_{\mathcal{H}_{\text{out}}}(Y) = 0_{\mathcal{H}_{\text{in}}}$. Note that $\mathrm{Tr}_{\mathcal{H}_{\text{out}}}(X+Y) = I_{\mathcal{H}_{\text{in}}}$. Since $\mathcal{C}$ is trace-preserving-preserving we know from Claim 2 that

$$I_{\mathcal{K}_{\text{in}}} = \mathrm{Tr}_{\mathcal{K}_{\text{out}}}(K_\mathcal{C}(X+Y)) = \underbrace{\mathrm{Tr}_{\mathcal{K}_{\text{out}}}(K_\mathcal{C}(X))}_{I_{\mathcal{K}_{\text{in}}}} + \mathrm{Tr}_{\mathcal{K}_{\text{out}}}(K_\mathcal{C}(Y)), \tag{15}$$

from which it follows that $\mathrm{Tr}_{\mathcal{K}_{\text{out}}}(K_\mathcal{C}(Y)) = 0_{\mathcal{K}_{\text{in}}}$ as desired. □

Claim 4. *Suppose $\mathcal{C}$ is trace-preserving-preserving. Let $\{|i\rangle\langle j|\} \subset \mathrm{L}(\mathcal{H}_{\text{out}})$ and $\{|k\rangle\langle l|\} \subset \mathrm{L}(\mathcal{H}_{\text{in}})$ denote the standard bases for $\mathrm{L}(\mathcal{H}_{\text{out}})$ and $\mathrm{L}(\mathcal{H}_{\text{in}})$, respectively. For each k,l there exists a fixed $Q_{k,l} \in \mathrm{L}(\mathcal{K}_{\text{in}})$ such that for every i it holds that*

$$\mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_\mathcal{C}\left(|i\rangle\langle i| \otimes |k\rangle\langle l|\right)\right) = Q_{k,l}. \tag{16}$$

Proof. Let $X \in \mathrm{L}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$ with $\mathrm{Tr}_{\mathcal{H}_{\text{out}}}(X) = I_{\mathcal{H}_{\text{in}}}$. Pick any i, j and observe that

$$\mathrm{Tr}_{\mathcal{H}_{\text{out}}}\left(X + |i\rangle\langle i| \otimes |k\rangle\langle l| - |j\rangle\langle j| \otimes |k\rangle\langle l|\right) = I_{\mathcal{H}_{\text{in}}} + |k\rangle\langle l| - |k\rangle\langle l| = I_{\mathcal{H}_{\text{in}}} \tag{17}$$

Since $\mathcal{C}$ is trace-preserving-preserving we know from Claim 2 that

$$I_{\mathcal{K}_{\text{in}}} = \mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_{\mathcal{C}}\left(X + |i\rangle\langle i| \otimes |k\rangle\langle l| - |j\rangle\langle j| \otimes |k\rangle\langle l|\right)\right) \tag{18}$$

$$= \underbrace{\mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_{\mathcal{C}}(X)\right)}_{I_{\mathcal{K}_{\text{in}}}} + \mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_{\mathcal{C}}\left(|i\rangle\langle i| \otimes |k\rangle\langle l|\right)\right) - \mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_{\mathcal{C}}\left(|j\rangle\langle j| \otimes |k\rangle\langle l|\right)\right) \tag{19}$$

from which it follows that

$$\mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_{\mathcal{C}}\left(|i\rangle\langle i| \otimes |k\rangle\langle l|\right)\right) = \mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_{\mathcal{C}}\left(|j\rangle\langle j| \otimes |k\rangle\langle l|\right)\right) \tag{20}$$

as desired. □

Armed with these facts about $K_{\mathcal{C}}$, we are now ready to solve the problem.
Since $\mathcal{C}$ is completely completely positive, we know from Claim 1 that $K_{\mathcal{C}}$ is completely positive and hence $J(K_{\mathcal{C}})$ is positive semidefinite. It remains to verify the following conditions:

$$\mathrm{Tr}_{\mathcal{K}_{\text{out}}}(J(K_{\mathcal{C}})) = I_{\mathcal{H}_{\text{out}}} \otimes Q \tag{21}$$

$$\mathrm{Tr}_{\mathcal{K}_{\text{out}} \otimes \mathcal{H}_{\text{in}}}(J(K_{\mathcal{C}})) = I_{\mathcal{K}_{\text{in}} \otimes \mathcal{H}_{\text{out}}} \tag{22}$$

for some $Q \in \mathrm{L}(\mathcal{H}_{\text{in}} \otimes \mathcal{K}_{\text{in}})$.
We begin with (21). Let us expand the definition of $J(K_{\mathcal{C}})$ and apply the partial trace $\mathrm{Tr}_{\mathcal{K}_{\text{out}}}$:

$$\mathrm{Tr}_{\mathcal{K}_{\text{out}}}(J(K_{\mathcal{C}})) = \sum_{i,j,k,l} \mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_{\mathcal{C}}\left(|i\rangle\langle j| \otimes |k\rangle\langle l|\right)\right) \otimes |i\rangle\langle j| \otimes |k\rangle\langle l|. \tag{23}$$

For each i, j, k, l we have

$$\mathrm{Tr}_{\mathcal{H}_{\text{out}}}(|i\rangle\langle j| \otimes |k\rangle\langle l|) = \langle i||j\rangle|k\rangle\langle l| = \delta_{i,j}|k\rangle\langle l| \tag{24}$$

When $i \neq j$ this expression vanishes. In this case, we know from Claim 3 that

$$\mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_{\mathcal{C}}\left(|i\rangle\langle j| \otimes |k\rangle\langle l|\right)\right) = 0_{\mathcal{K}_{\text{in}}} \tag{25}$$

Thus, the summands of (23) vanish whenever $i \neq j$.
We know from Claim 4 that for each k, l there exists $Q_{k,l}$ such that for all i we have

$$\mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_{\mathcal{C}}\left(|i\rangle\langle i| \otimes |k\rangle\langle l|\right)\right) = Q_{k,l}. \tag{26}$$

Substituting (25) and (26) into (23) we find that

$$\mathrm{Tr}_{\mathcal{K}_{\text{out}}}(J(K_{\mathcal{C}})) = \sum_{i,k,l} \mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_{\mathcal{C}}\left(|i\rangle\langle i| \otimes |k\rangle\langle l|\right)\right) \otimes |i\rangle\langle i| \otimes |k\rangle\langle l| \tag{27}$$

$$= \sum_{k,l} Q_{k,l} \otimes \left(\sum_i |i\rangle\langle i|\right) \otimes |k\rangle\langle l| \tag{28}$$

$$= I_{\mathcal{H}_{\text{out}}} \otimes \underbrace{\left(\sum_{k,l} Q_{k,l} \otimes |k\rangle\langle l|\right)}_{Q \in \mathrm{L}(\mathcal{H}_{\text{in}} \otimes \mathcal{K}_{\text{in}})} \tag{29}$$

Eq. (21) is thus established.

In order to verify (22) we apply the partial trace $\mathrm{Tr}_{\mathcal{H}_{\text{in}}}$ to (27):

$$\mathrm{Tr}_{\mathcal{K}_{\text{out}}\otimes\mathcal{H}_{\text{in}}}(J(K_{\mathcal{C}})) = \sum_{i,k,l} \mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_{\mathcal{C}}\left(|i\rangle\langle i| \otimes |k\rangle\langle l|\right)\right) \otimes |i\rangle\langle i| \otimes \underbrace{\mathrm{Tr}(|k\rangle\langle l|)}_{\delta_{k,l}} \tag{30}$$

$$= \sum_{i} \mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_{\mathcal{C}}\left(|i\rangle\langle i| \otimes \left(\sum_k |k\rangle\langle k|\right)\right)\right) \otimes |i\rangle\langle i| \tag{31}$$

$$= \sum_{i} \mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_{\mathcal{C}}\left(|i\rangle\langle i| \otimes I_{\mathcal{H}_{\text{in}}}\right)\right) \otimes |i\rangle\langle i| \tag{32}$$

For each i it clearly holds that $\mathrm{Tr}_{\mathcal{H}_{\text{out}}}(|i\rangle\langle i| \otimes I_{\mathcal{H}_{\text{in}}}) = I_{\mathcal{H}_{\text{in}}}$. Since $\mathcal{C}$ is trace-preserving-preserving we know from Claim 2 that

$$\mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(K_{\mathcal{C}}\left(|i\rangle\langle i| \otimes I_{\mathcal{H}_{\text{in}}}\right)\right) = I_{\mathcal{K}_{\text{in}}}. \tag{33}$$

Substituting (33) into (32) we obtain

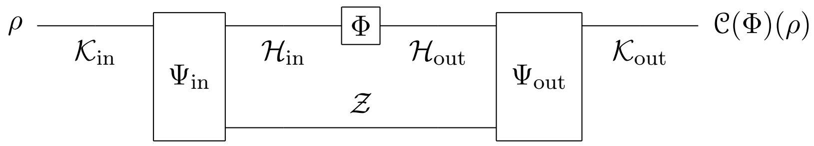
$$\mathrm{Tr}_{\mathcal{K}_{\text{out}}\otimes\mathcal{H}_{\text{in}}}(J(K_{\mathcal{C}})) = I_{\mathcal{K}_{\text{in}}} \otimes \left(\sum_i |i\rangle\langle i|\right) = I_{\mathcal{K}_{\text{in}}\otimes\mathcal{H}_{\text{out}}} \tag{34}$$

as desired.

(b) *[5 marks.]* Use the link product to prove that every channel-channel $\mathcal{C}$ admits a physical implementation whereby there exists a memory space $\mathcal{Z}$ and channels

$$\Psi_{\text{in}} : \mathrm{L}(\mathcal{K}_{\text{in}}) \to \mathrm{L}(\mathcal{H}_{\text{in}} \otimes \mathcal{Z})$$
$$\Psi_{\text{out}} : \mathrm{L}(\mathcal{H}_{\text{out}} \otimes \mathcal{Z}) \to \mathrm{L}(\mathcal{K}_{\text{out}})$$

such that for every channel $\Phi : \mathrm{L}(\mathcal{H}_{\text{in}}) \to \mathrm{L}(\mathcal{H}_{\text{out}})$ and every input state $\rho \in \mathrm{Dens}(\mathcal{K}_{\text{in}})$ it holds that the output state $\mathcal{C}(\Phi)(\rho) \in \mathrm{Dens}(\mathcal{K}_{\text{out}})$ is given by the following circuit.



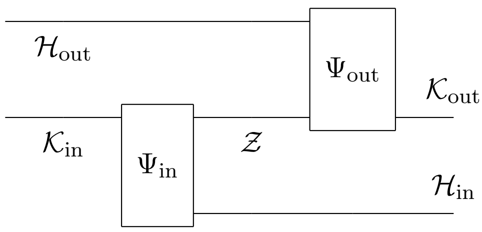
(35)

Solution. By definition we have

$$J(\Phi) * J(K_{\mathcal{C}}) = \mathrm{Tr}_{\mathcal{H}_{\text{out}}\otimes\mathcal{H}_{\text{in}}}((I_{\mathcal{K}_{\text{out}}\otimes\mathcal{K}_{\text{in}}} \otimes J(\Phi)^{\mathsf{T}})J(K_{\mathcal{C}})) = K_{\mathcal{C}}(J(\Phi)) = J(\mathcal{C}(\Phi)). \tag{36}$$

So it remains to prove that the channel whose Choi matrix is given by $J(\Phi) * J(K_{\mathcal{C}})$ is given by a circuit of the form (35).

We know from question 4a that $J(K_{\mathcal{C}})$ is a two-round strategy for input spaces $\mathcal{K}_{\text{in}}, \mathcal{H}_{\text{out}}$ and output spaces $\mathcal{H}_{\text{in}}, \mathcal{K}_{\text{out}}$. Thus, there exists a memory space $\mathcal{Z}$ and channels $\Psi_{\text{in}}, \Psi_{\text{out}}$ as in the statement of the question such that the super-operator $K_{\mathcal{C}}$ has the form



(37)

We know from the properties of the link product that $J(\Phi) * J(K_{\mathcal{C}})$ is the Choi matrix for the channel whose circuit is given by (35).

(c) *[5 marks.]* Prove the converse of question 4a: if $S \in \mathrm{L}(\mathcal{K}_{\text{out}} \otimes \mathcal{K}_{\text{in}} \otimes \mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$ is a two-round strategy for input spaces $\mathcal{K}_{\text{in}}, \mathcal{H}_{\text{out}}$ and output spaces $\mathcal{H}_{\text{in}}, \mathcal{K}_{\text{out}}$ then $S = J(K_{\mathcal{C}})$ for some channel-channel $\mathcal{C}$.

Solution. We must argue that $\mathcal{C}$ is completely completely positive and trace-preserving-preserving. To this end let $\mathcal{X}_{\text{in}}, \mathcal{X}_{\text{out}}$ be arbitrary spaces and let $\Psi : \mathrm{L}(\mathcal{H}_{\text{in}} \otimes \mathcal{X}_{\text{in}}) \to \mathrm{L}(\mathcal{H}_{\text{out}} \otimes \mathcal{X}_{\text{out}})$ be an arbitrary completely positive super-operator. We have

$$J\left(\left(\mathcal{C} \otimes \mathbb{1}_{\mathrm{T}(\mathcal{X}_{\text{in}},\mathcal{X}_{\text{out}})}\right)(\Psi)\right) = K_{\mathcal{C} \otimes \mathbb{1}_{\mathrm{T}(\mathcal{X}_{\text{in}},\mathcal{X}_{\text{out}})}}(J(\Psi)) = \left(K_{\mathcal{C}} \otimes \mathbb{1}_{\mathcal{X}_{\text{out}} \otimes \mathcal{X}_{\text{in}}}\right)(J(\Psi)). \tag{38}$$

Since $S = J(K_{\mathcal{C}})$ is positive semidefinite, we know that $K_{\mathcal{C}}$ is completely positive, from which it follows that the right side of (38) is positive semidefinite. Hence $\left(\mathcal{C} \otimes \mathbb{1}_{\mathrm{T}(\mathcal{X}_{\text{in}},\mathcal{X}_{\text{out}})}\right)(\Psi)$ is completely positive and so $\mathcal{C}$ is completely completely positive.

To see that $\mathcal{C}$ is trace-preserving-preserving let $\Phi : \mathrm{L}(\mathcal{H}_{\text{in}}) \to \mathrm{L}(\mathcal{H}_{\text{out}})$ be an arbitrary trace-preserving super-operator. We must show that $\mathrm{Tr}_{\mathcal{K}_{\text{out}}}(J(\mathcal{C}(\Phi))) = I_{\mathcal{K}_{\text{in}}}$. Since $S = J(K_{\mathcal{C}})$ is a two-round strategy we have

$$\mathrm{Tr}_{\mathcal{K}_{\text{out}}}(J(K_{\mathcal{C}})) = I_{\mathcal{H}_{\text{out}}} \otimes Q \tag{39}$$
$$\mathrm{Tr}_{\mathcal{H}_{\text{in}}}(Q) = I_{\mathcal{K}_{\text{in}}} \tag{40}$$

for some $Q \in \mathrm{L}(\mathcal{H}_{\text{in}} \otimes \mathcal{K}_{\text{in}})$. We have

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{K}_{\text{out}}}(J(\mathcal{C}(\Phi))) &= \mathrm{Tr}_{\mathcal{K}_{\text{out}}}\left(\mathrm{Tr}_{\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}}((I_{\mathcal{K}_{\text{out}} \otimes \mathcal{K}_{\text{in}}} \otimes J(\Phi)^{\mathsf{T}})J(K_{\mathcal{C}}))\right) && (41)\\
&= \mathrm{Tr}_{\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}}((I_{\mathcal{K}_{\text{in}}} \otimes J(\Phi)^{\mathsf{T}})\underbrace{\mathrm{Tr}_{\mathcal{K}_{\text{out}}}(J(K_{\mathcal{C}}))}_{I_{\mathcal{H}_{\text{out}}} \otimes Q}) && (42)\\
&= \mathrm{Tr}_{\mathcal{H}_{\text{in}}}((I_{\mathcal{K}_{\text{in}}} \otimes \underbrace{\mathrm{Tr}_{\mathcal{H}_{\text{out}}}(J(\Phi)^{\mathsf{T}})}_{I_{\mathcal{H}_{\text{in}}}})Q) && (43)\\
&= \mathrm{Tr}_{\mathcal{H}_{\text{in}}}(Q) = I_{\mathcal{K}_{\text{in}}} && (44)
\end{aligned}$$

as desired.