Lecture 2: The link product, combinations of strategies, output probabilities

We would like to define a bilinear product on Choi matrices that combines two strategies Q, R into one $Q * R$ by "hooking up" message spaces in the most natural way.

1 Motivating examples

Example 1 (General goal)**.** Let Q be a 4-round strategy for input spaces $\mathcal{X}_1, \ldots, \mathcal{X}_4$ and output spaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_4$.

[*** Draw a 4-round strategy. ***]

Let R be a 2-round strategy for input spaces $\mathcal{Y}_1, \mathcal{Y}_2$ and output spaces $\mathcal{X}_2, \mathcal{X}_3$.

[*** Draw a 2-round strategy. ***]

We want to define "$*$" so that $Q * R$ is the 2-round strategy for input spaces $\mathcal{X}_1, \mathcal{X}_4$ and output spaces $\mathcal{Y}_3, \mathcal{Y}_4$ obtained by composing the channels that define Q and R over the spaces shared between them.

[*** Draw the composed strategy. This makes more sense if you can see the picture. I did not make a new graphic in these notes. ***]

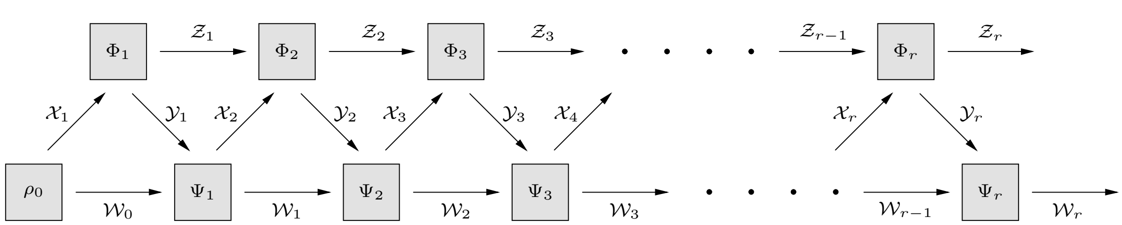
□

Example 2 (General goal, with measurement)**.** Same as above, except replace the non-measuring strategy Q with a measuring strategy $\{Q_a\}$. We want $\{Q_a * R\}$ to be the measuring strategy obtained by bundling $\{Q_a\}$ with R. □

Example 3 (Special case, output probabilities)**.** Let $\{Q_a\}$ be a r-round measuring strategy for input spaces $\mathcal{X}_1, \ldots, \mathcal{X}_r$ and output spaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_r$.

Let R be a $(r+1)$-round non-measuring strategy for input spaces $\mathbb{C}, \mathcal{Y}_1, \ldots, \mathcal{Y}_r$ and output spaces $\mathcal{X}_1, \ldots, \mathcal{X}_r, \mathbb{C}$.

Bundling Q_a with R leaves no free spaces, so we want $Q_a * R$ to be a scalar—the probability with which an interaction between $\{Q_a\}$ and R produces outcome a. (Like in the following diagram except with the measurement operator P_a applied to $\mathcal{Z}_r$ and the system $\mathcal{W}_r$ traced out.)



Similarly, if R is replaced with a measuring strategy $\{R_b\}$ then $Q_a * R_b$ should be the probability with which the interaction yields outcomes (a, b). Of course, if both Q and R are non-measuring strategies then $Q * R$ should be the scalar 1, corresponding to the intuition that non-measuring strategies can be viewed as measuring strategies with only one outcome, which must occur with probability 1. □

Example 4. If $\Phi : \mathrm{L}(\mathcal{W}) \to \mathrm{L}(\mathcal{X})$ and $\Psi : \mathrm{L}(\mathcal{X}) \to \mathrm{L}(\mathcal{Y})$ then want $J(\Phi) * J(\Psi) = J(\Psi \circ \Phi)$. □

The definition of the link product will exploit a key property of the vec mapping we saw in the first lecture: for $A : \mathcal{W} \to \mathcal{X}$ and $B : \mathcal{X} \to \mathcal{Y}$ it holds that

$$(\mathrm{vec}(I_\mathcal{X})^* \otimes I_{\mathcal{W}\otimes\mathcal{Y}}) \, \mathrm{vec}(A \otimes B) = \mathrm{vec}(BA). \tag{1}$$

(For brevity we often omit the tensor with identity.)

How does (1) help us? Consider Example 4. Write

$$\Phi : W \mapsto \sum_i A_i W A_i^* \qquad \Psi : X \mapsto \sum_j B_j X B_j^*$$

so that

$$\Psi \circ \Phi : W \mapsto \sum_{i,j} (B_j A_i) W (B_j A_i)^*.$$

We know that

$$J(\Phi) = \sum_i \mathrm{vec}(A_i) \, \mathrm{vec}(A_i)^* \qquad J(\Psi) = \sum_j \mathrm{vec}(B_j) \, \mathrm{vec}(B_j)^*$$

and so

$$J(\Psi \circ \Phi) = \sum_{i,j} \mathrm{vec}(B_j A_i) \, \mathrm{vec}(B_j A_i)^*.$$

Employing property (1) of the vec mapping, it holds that

$$J(\Psi \circ \Phi) = \mathrm{vec}(I_\mathcal{X})^* \, (J(\Psi) \otimes J(\Phi)) \, \mathrm{vec}(I_\mathcal{X})$$

so it makes sense to define the link product to satisfy the above identity.

We will also use a generalization of (1) wherein the output space of A agrees only partially with the input space of B: for $A : \mathcal{W} \to \mathcal{X} \otimes \mathcal{A}$ and $B : \mathcal{X} \otimes \mathcal{B} \to \mathcal{Y}$ it holds that

$$(\mathrm{vec}(I_\mathcal{X})^* \otimes I_{\mathcal{W}\otimes\mathcal{Y}\otimes\mathcal{A}\otimes\mathcal{B}}) \, \mathrm{vec}(A \otimes B) = \mathrm{vec}((B \otimes I_\mathcal{A})(A \otimes I_\mathcal{B})). \tag{2}$$

2 Definition of the link product

Definition 5 (Link product)**.** For any spaces $\mathcal{W}, \mathcal{X}, \mathcal{Y}$ and any operators $Q \in \mathrm{L}(\mathcal{X} \otimes \mathcal{W})$, $R \in \mathrm{L}(\mathcal{Y} \otimes \mathcal{X})$ the *link product* $Q * R \in \mathrm{L}(\mathcal{Y} \otimes \mathcal{W})$ of Q and R is a bilinear form defined by

$$Q * R \stackrel{\text{def}}{=} \mathrm{vec}(I_\mathcal{X})^* \, (Q \otimes R) \, \mathrm{vec}(I_\mathcal{X})$$

□

Example 6. If $\dim(\mathcal{Y}) = \dim(\mathcal{W}) = 1$ then $A * B = \mathrm{Tr}(A^\mathsf{T} B) = \mathrm{Tr}(A B^\mathsf{T})$. □

Example 7. If $\dim(\mathcal{X}) = 1$ then $A * B = A \otimes B$. □

Example 3 is a very important special case of the link product, so let us examine this example in detail. It should become clear how the following analysis extends to other, more general examples.

Recall that $\{Q_a\}$ is an r-round measuring strategy for input spaces $\mathcal{X}_1, \ldots, \mathcal{X}_r$ and output spaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_r$. Let $\mathcal{Z}_1, \ldots, \mathcal{Z}_r$ be memory spaces and let $(\Phi_1, \ldots, \Phi_r, \{P_a\})$ be a description of the channels and measurement from which $\{Q_a\}$ is derived.

As mentioned previously, we may assume the channels are isometries: $\Phi_i : X \mapsto A_i X A_i^*$ for isometries

$$\begin{aligned} A_1 &: \mathcal{X}_1 \to \mathcal{Y}_1 \otimes \mathcal{Z}_1 \\ A_i &: \mathcal{X}_i \otimes \mathcal{Z}_{i-1} \to \mathcal{Y}_i \otimes \mathcal{Z}_i \qquad i = 2, \ldots, r \end{aligned}$$

As mentioned previously, let $A : \mathcal{X}_{1\ldots r} \to \mathcal{Y}_{1\ldots r} \otimes \mathcal{Z}_r$ be the isometry $A = A_r \cdots A_1$ obtained by composing $A_1, \ldots, A_r$ on the memory spaces $\mathcal{Z}_1, \ldots, \mathcal{Z}_{r-1}$ so that each $Q_a = J(\Xi_a)$ for

$$\Xi_a : X \mapsto \operatorname{Tr}_{\mathcal{Z}_r}(P_a A X A^*).$$

That is, $Q_a = \operatorname{Tr}_{\mathcal{Z}_r}(P_a \operatorname{vec}(A) \operatorname{vec}(A)^*)$.

Employing the property (2) of the vec mapping, we may write

$$\operatorname{vec}(A) = \operatorname{vec}(I_{\mathcal{Z}_{1\ldots r-1}})^* \left(\operatorname{vec}(A_r) \otimes \cdots \otimes \operatorname{vec}(A_1)\right).$$

Similarly, recall that R is an $(r+1)$-round non-measuring strategy for input spaces $\mathbb{C}, \mathcal{Y}_1, \ldots, \mathcal{Y}_r$ and output spaces $\mathcal{X}_1, \ldots, \mathcal{X}_r, \mathbb{C}$. Let $\mathcal{W}_0, \ldots, \mathcal{W}_r$ be memory spaces and let $(\rho_0, \Psi_1, \ldots, \Psi_r)$ be a description of the channels from which R is derived. (Recall that the first input space for this strategy is one-dimensional, so the first "channel" is actually just a state ρ_0.)

As above, we may assume that the state $\rho_0 = |\psi\rangle\langle\psi|$ is pure for some $|\psi\rangle \in \mathcal{X}_1 \otimes \mathcal{W}_0$ and that the channels are isometries: $\Psi_i : Y \mapsto B_i Y B_i^*$ for isometries

$$\begin{aligned} B_i &: \mathcal{Y}_i \otimes \mathcal{W}_{i-1} \to \mathcal{X}_{i+1} \otimes \mathcal{W}_i \qquad i = 1, \ldots, r-1 \\ B_r &: \mathcal{Y}_r \otimes \mathcal{W}_{r-1} \to \mathcal{W}_r \end{aligned}$$

As above, let $B : \mathcal{Y}_{1\ldots r} \to \mathcal{X}_{1\ldots r} \otimes \mathcal{W}_r$ be the isometry $B = B_r \cdots B_1 |\psi\rangle$ obtained by composing $|\psi\rangle, B_1, \ldots, B_r$ on the memory spaces $\mathcal{W}_0, \ldots, \mathcal{W}_{r-1}$ so that $R = J(\Upsilon)$ for

$$\Upsilon : Y \mapsto \operatorname{Tr}_{\mathcal{W}_r}(B Y B^*).$$

That is, $R = \operatorname{Tr}_{\mathcal{W}_r}(\operatorname{vec}(B) \operatorname{vec}(B)^*)$.

Again, employing the property (2) of the vec mapping we may write

$$\operatorname{vec}(B) = \operatorname{vec}(I_{\mathcal{W}_{0\ldots r-1}})^* \left(\operatorname{vec}(B_r) \otimes \cdots \otimes \operatorname{vec}(B_1) \otimes |\psi\rangle\right).$$

Employ (2) once again to observe that

$$\begin{aligned} &\operatorname{vec}(I_{\mathcal{Y}_{1\ldots r} \otimes \mathcal{X}_{1\ldots r}})^* \operatorname{vec}(A) \otimes \operatorname{vec}(B) \\ =\ &\operatorname{vec}(I_{\mathcal{Y}_{1\ldots r} \otimes \mathcal{X}_{1\ldots r} \otimes \mathcal{Z}_{1\ldots r-1} \otimes \mathcal{W}_{0\ldots r-1}})^* \left(\operatorname{vec}(B_r) \otimes \cdots \otimes \operatorname{vec}(B_1) \otimes |\psi\rangle \otimes \operatorname{vec}(A_r) \otimes \cdots \otimes \operatorname{vec}(A_1)\right) \\ =\ &\operatorname{vec}(B_r A_r \cdots B_1 A_1 |\psi\rangle) \\ =\ &B_r A_r \cdots B_1 A_1 |\psi\rangle \stackrel{\text{def}}{=} |\phi\rangle \in \mathcal{Z}_r \otimes \mathcal{W}_r \end{aligned}$$

Finally, let us apply the definition of the link product to compute $Q_a * R$.

$$\begin{aligned} Q_a * R &= \mathrm{Tr}_{\mathcal{Z}_r \otimes \mathcal{W}_r} \left(\mathrm{vec}(I_{\mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r}})^* P_a \,\mathrm{vec}(A \otimes B)\, \mathrm{vec}(A \otimes B)^* \,\mathrm{vec}(I_{\mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r}}) \right) \\ &= \mathrm{Tr}_{\mathcal{Z}_r \otimes \mathcal{W}_r} \left(P_a |\phi\rangle\langle\phi| \right) = \mathrm{Tr}\left(P_a |\phi\rangle\langle\phi| \right) \\ &= \Pr[\{Q_a\} \text{ yields outcome } a \text{ when interacting with } R] \end{aligned}$$

as desired.

Applying Example 6 we have that the above probability is given by the simple formula $Q_a * R = \mathrm{Tr}(Q_a R^{\mathsf{T}})$.

3 Sloppiness

Be careful! We are being sloppy here.

Recall Example 3: $\{Q_a\}$ is an r-round measuring strategy for input spaces $\mathcal{X}_1, \dots, \mathcal{X}_r$ and output spaces $\mathcal{Y}_1, \dots, \mathcal{Y}_r$. R is an $(r+1)$-round non-measuring strategy for input spaces $\mathbb{C}, \mathcal{Y}_1, \dots, \mathcal{Y}_r$ and output spaces $\mathcal{X}_1, \dots, \mathcal{X}_r, \mathbb{C}$. As such R and each Q_a has the form

$$\begin{aligned} Q_a &\in \mathrm{Pos}(\mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r}) \\ R &\in \mathrm{Pos}(\mathcal{X}_{1\dots r} \otimes \mathcal{Y}_{1\dots r}). \end{aligned}$$

There is a type-mismatch here. Strictly speaking, the formula

$$Q_a * R = \mathrm{vec}(I_{\mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r}})^* (Q_a \otimes R) \,\mathrm{vec}(I_{\mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r}})$$

is not defined because the spaces don't line up.

If you told a computer to compute this quantity, the quantity you obtain would *not* be the probability of outcome a!

In detail, $\mathrm{vec}(I_{\mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r}})$ is an element of

$$\mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r} \otimes \mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r}.$$

So in order for conjugation by $\mathrm{vec}(I_{\mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r}})$ to be defined $Q_a \otimes R$ needs to be an element of

$$\mathrm{L}(\mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r} \otimes \mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r}).$$

But instead it's an element of

$$\mathrm{L}(\mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r} \otimes \mathcal{X}_{1\dots r} \otimes \mathcal{Y}_{1\dots r}).$$

When combining two strategies via the link product, we need to ensure that the spaces align appropriately before conjugating by $\mathrm{vec}(I)$.

In the case of Example 3, we need to replace R with WRW^* where

$$W : \mathcal{X}_{1\dots r} \otimes \mathcal{Y}_{1\dots r} \to \mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r} : |i\rangle|j\rangle \mapsto |j\rangle|i\rangle$$

is the unitary that swaps $\mathcal{X}_{1\dots r}$ with $\mathcal{Y}_{1\dots r}$.

In general, different instances of the link product $A * B$ require a different swap/permutation unitary. Exactly which swap unitary depends upon the ordering of the spaces on which A, B act. Any such swap unitary that properly aligns the spaces prior to conjugation by $\operatorname{vec}(I)$ will suffice.

Thus, the probability of outcome a in Example 3 is correctly written

$$Q_a * R = \operatorname{Tr}(Q_a W R^{\mathsf{T}} W^*).$$

4 Co-strategies

Interactions between two strategies that do not leave any free spaces when combined arise frequently in various studies. It is worth refining terminology to make things simpler.

Recall Example 3: in order to leave no free spaces, an r-round strategy Q (measuring or no) for input spaces $\mathcal{X}_1, \ldots, \mathcal{X}_r$ and output spaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_r$ must interact with an $(r+1)$-round strategy R for input spaces $\mathbb{C}, \mathcal{Y}_1, \ldots, \mathcal{Y}_r$ and output spaces $\mathcal{X}_1, \ldots, \mathcal{X}_r, \mathbb{C}$.

Definition 8 (Co-strategy)**.** Let R (or $\{R_b\}$) be an $(r+1)$-round (measuring) strategy for input spaces $\mathbb{C}, \mathcal{Y}_1, \ldots, \mathcal{Y}_r$ and output spaces $\mathcal{X}_1, \ldots, \mathcal{X}_r, \mathbb{C}$. Then $W R^{\mathsf{T}} W^*$ (or $\{W R_b^{\mathsf{T}} W^*\}$) is called an r-round (measuring) *co-strategy* for input spaces $\mathcal{X}_1, \ldots, \mathcal{X}_r$ and output spaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_r$. □

In particular, if $\{Q_a\}$ is a measuring strategy and $\{R_b\}$ is a measuring co-strategy for the same input and output spaces then the probability of outcomes (a, b) is given by the simple bilinear formula $\operatorname{Tr}(Q_a R_b)$.

5 Alternate formula

For arbitrary $Q \in \mathrm{L}(\mathcal{W} \otimes \mathcal{X})$ and $R \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Y})$ the link product is also given by

$$Q * R = \operatorname{Tr}_{\mathcal{X}} \left((I_{\mathcal{Y}} \otimes Q) \left(R^{\mathsf{T}_{\mathcal{X}}} \otimes I_{\mathcal{W}} \right) \right) = \operatorname{Tr}_{\mathcal{X}} \left(\left(I_{\mathcal{Y}} \otimes Q^{\mathsf{T}_{\mathcal{X}}} \right) (R \otimes I_{\mathcal{W}}) \right)$$

where $\mathsf{T}_{\mathcal{X}}$ denotes the partial transpose on $\mathcal{X}$.

As a special case we recover the formula for obtaining the action of a channel $\Phi : \mathrm{L}(\mathcal{X}) \to \mathrm{L}(\mathcal{Y})$ on a state $\rho \in \mathrm{Dens}(\mathcal{X})$ that we saw at the beginning of the previous lecture:

$$\Phi(\rho) = J(\Phi(\rho)) = J(\Phi) * J(\rho) = J(\Phi) * \rho = \operatorname{Tr}_{\mathcal{X}} \left(\left(I_{\mathcal{Y}} \otimes \rho^{\mathsf{T}} \right) J(\Phi) \right) = \operatorname{Tr}_{\mathcal{X}} \left((I_{\mathcal{Y}} \otimes \rho) \, J(\Phi)^{\mathsf{T}_{\mathcal{X}}} \right).$$