Lecture 3: Characterization of strategies

In the previous lecture we saw how to combine two strategies into one via the link product, including as a special case combining a strategy and co-strategy to yield a simple bilinear formula for the probability of obtaining a given measurement outcome.

In this lecture we address the question, "Given an operator Q, how can we tell whether Q describes a legal strategy?" These two pieces of formalism (the link product and the characterization of strategies) opens the door to various optimization tasks for strategies. One example we will see is the min-max theorem for zero-sum quantum games.

1 Characterization of strategies

Theorem 1 (Characterization of non-measuring strategies)**.** *$Q \in \mathrm{Pos}(\mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r})$ is an r-round non-measuring strategy for input spaces $\mathcal{X}_1, \dots, \mathcal{X}_r$ and output spaces $\mathcal{Y}_1, \dots, \mathcal{Y}_r$ if and only if there exists an $(r-1)$-round non-measuring strategy $R \in \mathrm{Pos}(\mathcal{Y}_{1\dots r-1} \otimes \mathcal{X}_{1\dots r-1})$ for input spaces $\mathcal{X}_1, \dots, \mathcal{X}_{r-1}$ and output spaces $\mathcal{Y}_1, \dots, \mathcal{Y}_{r-1}$ with*

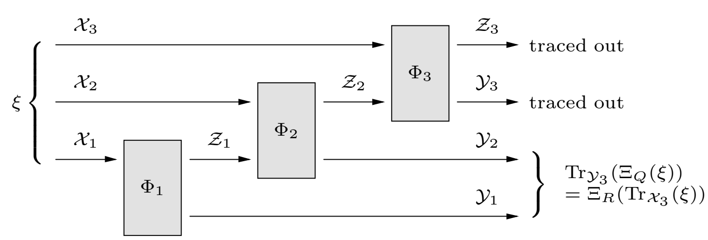
$$\mathrm{Tr}_{\mathcal{Y}_r}(Q) = R \otimes I_{\mathcal{X}_r}$$

Proof ($\Longrightarrow$). Assume Q is an r-round strategy. We must show that $\mathrm{Tr}_{\mathcal{Y}_r}(Q) = R \otimes I_{\mathcal{X}_r}$ for some $(r-1)$-round strategy R.

To this end let $\Phi_1, \dots, \Phi_r$ be the channels comprising the r-round strategy Q and let $\Xi_Q = \Phi_r \circ \cdots \circ \Phi_1$ be the channel with $J(\Xi_Q) = Q$.

Let $\Xi_R = \Phi_{r-1} \circ \cdots \circ \Phi_1$ and let $R = J(\Xi_R)$ be the $(r-1)$-round strategy obtained by truncating Q after only $r-1$ rounds.

It is clear from the following figure that $(\mathrm{Tr}_{\mathcal{Y}_r} \circ \Xi_Q) = (\Xi_R \otimes \mathrm{Tr}_{\mathcal{X}_r})$.



We have

$$\mathrm{Tr}_{\mathcal{Y}_r}(Q) = \mathrm{Tr}_{\mathcal{Y}_r}(J(\Xi_Q)) = J(\mathrm{Tr}_{\mathcal{Y}_r} \circ \Xi_Q) = J(\Xi_R \otimes \mathrm{Tr}_{\mathcal{X}_r}) = R \otimes I_{\mathcal{X}_r}$$

as desired.

(The second equality follows from the simple observation that any super-operator acting only on output spaces [such as the partial trace $\mathrm{Tr}_{\mathcal{Y}_r}$] can slide inside the Choi-Jamiołkowski mapping J. This observation can be proved easily and directly from the definition of the Choi-Jamiołkowski isomorphism.) □

Proof ($\Longleftarrow$). Assume $\mathrm{Tr}_{\mathcal{Y}_r}(Q) = R \otimes I_{\mathcal{X}_r}$ for some $(r-1)$-round strategy R. We must show that Q is an r-round strategy.

Since R is an $(r-1)$-round strategy we may select isometries $A_1, \ldots, A_{r-1}$ describing R and let $A_R = A_{r-1} \cdots A_1$ be the isometry obtained by composing $A_1, \ldots, A_{r-1}$ on memory spaces as usual so that

$$R = \mathrm{Tr}_{\mathcal{Z}_{r-1}}(\mathrm{vec}(A_R)\,\mathrm{vec}(A_R)^*).$$

Notice that $\mathrm{vec}(A_R)$ is a "purification" of R.

We claim that $\mathrm{vec}(I_{\mathcal{X}_r})$ is a purification of $I_{\mathcal{X}_r}$. Perhaps the easiest way to see this using facts we have already seen is to note that for any space $\mathcal{X}$ we have

$$\mathrm{Tr}_{\mathcal{X}}(\mathrm{vec}(I_{\mathcal{X}})\,\mathrm{vec}(I_{\mathcal{X}})^*) = \mathrm{Tr}_{\mathcal{X}}(J(\mathbb{1}_{\mathcal{X}})) = J(\mathrm{Tr}_{\mathcal{X}} \circ \mathbb{1}_{\mathcal{X}}) = J(\mathrm{Tr}_{\mathcal{X}}) = I_{\mathcal{X}}.$$

(Here $\mathbb{1}_{\mathcal{X}}$ denotes the identity channel on $\mathrm{L}(\mathcal{X})$.) Thus $\mathrm{vec}(A_R \otimes I_{\mathcal{X}_r})$ is a purification of $R \otimes I_{\mathcal{X}_r}$.

Let $\mathcal{Z}_r$ be a space with $\dim(\mathcal{Z}_r) = \max\{\mathrm{rank}(Q), \dim(\mathcal{Z}_{r-1} \otimes \mathcal{X}_r)/\dim(\mathcal{Y}_r)\}$.

Because $\dim(\mathcal{Z}_r) \geq \mathrm{rank}(Q)$, the space $\mathcal{Z}_r$ is large enough to admit a "purification" of Q. In particular, there exists an operator $B : \mathcal{X}_{1\ldots r} \to \mathcal{Y}_{1\ldots r} \otimes \mathcal{Z}_r$ with

$$\mathrm{Tr}_{\mathcal{Z}_r}(\mathrm{vec}(B)\,\mathrm{vec}(B)^*) = Q.$$

But then $\mathrm{vec}(B)$ is also a purification of $R \otimes I_{\mathcal{X}_r}$:

$$\mathrm{Tr}_{\mathcal{Y}_r \otimes \mathcal{Z}_r}(\mathrm{vec}(B)\,\mathrm{vec}(B)^*) = \mathrm{Tr}_{\mathcal{Y}_r}(Q) = R \otimes I_{\mathcal{X}_r}.$$

Because $\dim(\mathcal{Y}_r \otimes \mathcal{Z}_r) \geq \dim(\mathcal{Z}_{r-1} \otimes \mathcal{X}_r)$, by the unitary equivalence of purifications $\exists$ isometry $A_r : \mathcal{Z}_{r-1} \otimes \mathcal{X}_r \to \mathcal{Y}_r \otimes \mathcal{Z}_r$ such that

$$\mathrm{vec}(B) = A_r\,\mathrm{vec}(A_R \otimes I_{\mathcal{X}_r}) = \mathrm{vec}(A_r A_R) = \mathrm{vec}(A_r A_{r-1} \cdots A_1).$$

We have therefore exhibited isometries $A_1, \ldots, A_r$ such that the composition $A = A_r \cdots A_1$ satisfies $Q = \mathrm{Tr}_{\mathcal{Z}_r}(\mathrm{vec}(A)\,\mathrm{vec}(A)^*)$, so it must be that Q is an r-round strategy. $\square$

The recursive formula of Theorem 1 can be written explicitly as follows: Q is a strategy if and only if there exist $Q_i \in \mathrm{Pos}(\mathcal{Y}_{1\ldots i} \otimes \mathcal{X}_{1\ldots i})$ for $i = 1, \ldots, r$ with $Q = Q_r$ and

$$\begin{aligned} \mathrm{Tr}_{\mathcal{Y}_i}(Q_i) &= I_{\mathcal{X}_i} \otimes Q_{i-1} \qquad i = 2, \ldots, r \\ \mathrm{Tr}_{\mathcal{Y}_1}(Q_1) &= I_{\mathcal{X}_1} \end{aligned}$$

These linear constraints are closed under transposition and swapping of spaces. Hence, Q is a r-round co-strategy for input spaces $\mathcal{X}_1, \ldots, \mathcal{X}_r$ and output spaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_r$ if and only if there exist $Q_i \in \mathrm{Pos}(\mathcal{Y}_{1\ldots i-1} \otimes \mathcal{X}_{1\ldots i})$ for $i = 1, \ldots, r$ with $Q = Q_r \otimes I_{\mathcal{Y}_r}$ and

$$\begin{aligned} \mathrm{Tr}_{\mathcal{X}_i}(Q_i) &= I_{\mathcal{Y}_{i-1}} \otimes Q_{i-1} \qquad i = 2, \ldots, r \\ \mathrm{Tr}_{\mathcal{X}_1}(Q_1) &= 1 \end{aligned}$$

2 Corollaries of the characterization of strategies

We also obtain a tight bound on the amount of memory workspace required to implement any strategy.

Corollary 1.1 (Bound on required memory)**.** *Every r-round strategy Q can be described by isometries in such a way that the final memory space $\mathcal{Z}_r$ has dimension no larger than* $\operatorname{rank}(Q)$.

Proof. Induction on r. The base case is a well-known fact about Stinespring representations of channels: if $\Phi : \mathrm{L}(\mathcal{X}) \to \mathrm{L}(\mathcal{Y})$ is a channel then $\exists$ isometry $A : \mathcal{X} \to \mathcal{Y} \otimes \mathcal{Z}$ with $\Phi : X \mapsto \mathrm{Tr}_{\mathcal{Z}}(AXA^*)$ such that $\dim(\mathcal{Z}) = \operatorname{rank}(J(\Phi))$.

For the inductive step, assume that the corollary holds for $(r-1)$-round strategies. We will use this assumption to show that the corollary holds for r-round strategies.

Returning to the proof of Theorem 1, in selecting the isometries $A_1, \dots, A_{r-1}$ comprising R we may further assume (by induction) that the dimension of the final memory space $\mathcal{Z}_{r-1}$ equals $\operatorname{rank}(R)$, which is the minimum required to admit a purification of R.

Since $\operatorname{rank}(I_{\mathcal{X}_r}) = \dim(\mathcal{X}_r)$ the minimum dimension required to admit a purification of $I_{\mathcal{X}_r}$ is $\dim(\mathcal{X}_r)$. Hence, the space $\mathcal{Z}_{r-1} \otimes \mathcal{X}_r$ has minimal dimension among all spaces that admit purifications of $R \otimes I_{\mathcal{X}_r}$.

In the proof of Theorem 1 we constructed a second purification $\operatorname{vec}(B)$ of $R \otimes I_{\mathcal{X}_r}$ with "garbage" space $\mathcal{Y}_r \otimes \mathcal{Z}_r$. The condition $\dim(\mathcal{Y}_r \otimes \mathcal{Z}_r) \geq \dim(\mathcal{Z}_{r-1} \otimes \mathcal{X}_r)$ is met automatically since $\mathcal{Z}_{r-1} \otimes \mathcal{X}_r$ has minimal dimension among all spaces that admit purifications of $R \otimes I_{\mathcal{X}_r}$.

In other words, by choosing $\dim(\mathcal{Z}_r) = \operatorname{rank}(Q)$, we automatically satisfy the condition $\dim(\mathcal{Z}_r) = \max\{\operatorname{rank}(Q), \dim(\mathcal{Z}_{r-1} \otimes \mathcal{X}_r)/\dim(\mathcal{Y}_r)\}$ used in the proof of Theorem 1 to choose the space $\mathcal{Z}_r$. □

Finally, Theorem 1 can be used to characterize *measuring* strategies.

Corollary 1.2 (Characterization of measuring strategies)**.** *A set $\{Q_a\} \subset \mathrm{Pos}(\mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r})$ is a measuring strategy if and only if $\sum_a Q_a = Q$ for some non-measuring strategy Q.*

Proof. $\implies$ is easy and does not require Theorem 1. Given $\{Q_a\}$ let $A_1, \dots, A_r, \{P_a\}$ be isometries and a measurement comprising $\{Q_a\}$. By linearity

$$\sum_a Q_a = \mathrm{Tr}_{\mathcal{Z}_r}\left(\left(\left(\sum_a P_a\right) \otimes I_{\mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r}}\right) \operatorname{vec}(A) \operatorname{vec}(A)^*\right) \tag{1}$$

$$= \mathrm{Tr}_{\mathcal{Z}_r}\left(\operatorname{vec}(A) \operatorname{vec}(A)^*\right), \tag{2}$$

$\impliedby$ is a question on the assignment that requires Theorem 1. □

3 Application: a semidefinite program for maximum output probabilities

As mentioned earlier, the link product and characterization of strategies allow us to apply known efficient algorithms for optimizing over strategies.

For example, suppose we are given an r-round measuring strategy $\{Q_a\}$ and we wish to know what is the maximum probability with which this measuring strategy can be made to produce a given outcome a by interacting with some r-round co-strategy.

In other words, we wish to solve the optimization problem

$$\begin{array}{rl} \text{maximize} & \Pr[\{Q_a\} \text{ produces outcome } a \text{ when interacting with } R] \\ \text{subject to} & R \text{ is a co-strategy} \end{array}$$

Employing the link product and Theorem 1, this optimization problem can be written

$$\begin{array}{rl} \text{maximize} & \operatorname{Tr}(Q_a R) \\ \text{subject to} & R = I_{\mathcal{Y}_r} \otimes R_r \\ & \operatorname{Tr}_{\mathcal{X}_r}(R_r) = I_{\mathcal{Y}_{r-1}} \otimes R_{r-1} \\ & \quad \vdots \\ & \operatorname{Tr}_{\mathcal{X}_2}(R_2) = I_{\mathcal{Y}_1} \otimes R_1 \\ & \operatorname{Tr}_{\mathcal{X}_1}(R_1) = 1 \\ & R_i \in \operatorname{Pos}(\mathcal{Y}_{1\dots i-1} \otimes \mathcal{X}_{1\dots i}) \quad (1 \leq i \leq r) \\ & R \in \operatorname{Pos}(\mathcal{Y}_{1\dots r} \otimes \mathcal{X}_{1\dots r}) \end{array}$$

This problem is an example of a semidefinite program (SDP). Semidefinite programs have been intensively studied. It is known that there exist efficient, classical, deterministic algorithms to solve SDPs. Thus, there is an efficient, classical, deterministic algorithm to compute the maximum probability with which an arbirary measuring strategy can be made to produce a given outcome.

For comparison, recall from Lecture 1 the observation that a naive optimization over the operational representation of quantum strategies is not efficient. In particular, for an interaction between a measuring strategy specified by $(\Phi_1, \dots, \Phi_r, \{P_a\})$ and a co-strategy specified by $(\rho_0, \Psi_1, \dots, \Psi_r)$ the final state of the system is $(\Psi_r \circ \Phi_r \circ \cdots \circ \Psi_1 \circ \Phi_1)(\rho_0)$ and so the probability of outcome a is

$$\operatorname{Tr}\left(P_a(\Psi_r \circ \Phi_r \circ \cdots \circ \Psi_1 \circ \Phi_1)(\rho_0)\right),$$

which is highly non-linear in $(\rho_0, \Psi_1, \dots, \Psi_r)$.