Summary of Quantum Algorithms for Simplicial Homology (and Topological Data Analysis)

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In the paper "Quantum algorithms for topological and geometric analysis of data" by Lloyd, Garnerone and Zanardi, the authors devised a quantum algorithm for computing persistent homology. In practice, computing persistent homology is essentially about computing simplicial homologies for finitely many simplicial complexes obtained from a data set. In this expository note, we outline this algorithm for computing simplicial homology.

Given a set of n data points and a small real number ϵ, we can construct a simplicial complex $S = S^\epsilon$ where each data point is a zero simplex and vice versa. We have a total space $\mathbb{C}^{2^n}$, where each zero simplex corresponds to a standard basis vector. The boundary of each $k-$simplex consists of a unique collection of $k-1$ simplices. For each $k-$simplex s_k, we designate it by the $n-$qubit basis vector $|s_k\rangle \in \mathbb{C}^{2^n}$. Let W_k denote the $\binom{n}{k+1}$ dimensional Hilbert space corresponding to all possible k simplices, and let H_k denote the subspace of W_k which is spanned by $\{|s_k\rangle | s_k \in S_k\}$. Finally let $H = \oplus_k H_k$.

Next we use Grover's quantum search algorithm to find all $|s_k\rangle \in S_k$ from the total space, and construct the $k-$simplex state $|\phi\rangle_k = \frac{1}{|S_k|}\Sigma_{s_k \in S_k}|s_k\rangle$. Grover's algorithm gives a quadratic speedup over classical methods.

For the topological analysis, the quantum phase algorithm is used, which gives an exponential speedup over classical methods. Once the $k-$simplex states $|\phi\rangle_k$ are constructed, the quantum phase algorithm allows us to decompose those states in terms of eigenvectors and eigenvalues of some matrices obtained from the boundary maps. More specifically, we embed each boundary map ∂_k into a Hermitian matrix B_k, and define the full Hermitian boundary map to be $B = B_1 \oplus \cdots B_n$. Note that $B^2 = \triangle_0 \oplus \cdots \oplus \triangle_n$, where $\triangle_k$ is the combinatorial Laplacian of the $k-$the simplicial complex. The quantum phase algorithm gives the eigenvalues and eigenvectors of the unitary operator $U = e^{iB}$. Then it would not be hard to get the eigenvalues and eigenvectors of B and B^2. The original quantum phase algorithm starts with actual eigenvectors of unitary operators. Perhaps in this case the authors plan to measure the $k-$simplex states multiple times, and they are counting on the fact that the inner product of a $k-$simplex state and each eigenvector of ∂_k is not too small.

Since we are working over the filed of complex numbers, the Betti numbers can be easily deduced from the dimensions of the kernels of all the boundary maps. For complex geometry, we have Hodge theory. So the kernels of combinatorial Laplacians at different dimensions give rise to all the homology groups.