Homework set 2, PHY 790, due Thursday, February 21, 2013

5. Thinking about the material we have covered so far (in the lectures, homework, or reading), write down a physics *question* which you would like to know the answer to. (Some examples: a simple question suggested by the lectures but not answered by them; a more extensive question you think might make a worthwhile research project; a confusion about the material that you'd like to clear up for yourself; a philosophical puzzle related to the material.)

6. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ where X, Y and Z are vector spaces and f and g are smooth functions. The *chain rule* says that $(g \circ f)'(x) = g'(y) \circ f'(x)$, for $y = f(x)$. Verify it, assuming that $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$, $Z = \mathbb{R}^k$, and using what you know about partial derivatives. Or if you can, prove it instead from the definition $\Delta y = f'(x) \cdot \Delta x + o(\Delta x)$ that was mentioned in class. (Of course you should also feel free to prove it in any other way that works.)

7. Let $T_P M$ be the space of tangent vectors to the manifold M at the point $P \in M$, where, as in class, a tangent vector $v \in T_P M$ is defined to be an equivalence class of parameterized curves. Prove that our definition of the sum $v + w$ of two vectors in $T_P M$ is consistent (independent of the chart used in the definition). Hint: it might help to prove first that $\eta_* = f'_{\eta\xi} \circ \xi_*$.

8. (a) Exhibit an atlas for the two-sphere S^2 (both the chart-maps ξ and the transition-maps $f_{\xi\eta}$). (b) What is the minimum number of charts you could have used for your atlas? (c) Explain why the standard spherical coordinates don't furnish a valid chart in the sense we have defined this.

9. Consider the 4-dimensional surface $S \subseteq \mathbb{M}^5$ defined by the equation $a = t(T - t)/T$, where T is a positive constant, $(x^0, x^1, x^2, x^3, x^4)$ are standard Cartesian coordinates on $\mathbb{M}^5$, $t = x^0$, and $a = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2}$ is radial distance in $\mathbb{M}^5$ from the t-axis, $x^1 = x^2 = x^3 = x^4 = 0$. We can regard S as a 4-dimensional spacetime with its metric inherited from that of $\mathbb{M}^5$, and as such it is an example of the type of homogeneous and isotropic (Friedmannian) cosmos we discussed in class.
(a) Clearly, each slice of our cosmos is a sphere of radius a. Recalling that we defined $d\tau$ as the proper time between "neighboring" slices, find $da/d\tau$ for every slice. (Notice that you can use t to parameterize the slices.) (b) Prove that the spacetime S has the geometry of a dust-filled spherical Friedmann cosmos (the equations of motion are satisfied with $\Lambda = 0$) and find the total rest-mass M in terms of the parameter T. (c) From the 5d perspective, how rapidly is this cosmos expanding near its initial singularity. Sketch a picture of S as embedded in $\mathbb{M}^5$.