Elements of General Relativity I/II, 2013 (Rafael D Sorkin)

MATHEMATICAL LANGUAGE

Logical and Set-theoretical Symbols

$\forall$ means *for all* (The symbol is an upside down 'A' standing for 'all'.)

$\exists$ means *there exists* (or "for some") (The symbol is a backwards 'E' for 'exists'.)

$\Rightarrow$ means *implies*

iff is short for *if and only if*

$\{a, b, \cdots, c\}$ is the *set* whose *elements* are $a, b, \cdots c$

$\in$ means *belongs to* or *is an element of*

$\cup$ means *union*

$\cap$ means *intersection*

$\subseteq$ *or* $\subset$ means *subset of*

$\setminus$ means "set difference" or "less"

The set-formation symbol $\{\cdots\}$ is most commonly used in constructions like

$$\mathbb{N} = \{n \in \mathbb{Z} \,|\, n \geq 0\} \ ,$$

which can be read as "$\mathbb{N}$ equals *the set of all* integers n *such that* $n \geq 0$." Using this notation, we can define set-difference, "A less B", by

$$A \backslash B = \{x \in A \,|\, x \notin B\} \ .$$

Examples The set of even natural numbers is $\{0, 2, 4, 6, \cdots\} = 2\mathbb{N}$. We have $2 \in \{0, 2, 4\}$, $3 \notin \{0, 2, 4\}$. If x, y are positive real numbers and n a positive integer then "Archimedes' principle" says

$$(\forall x)(\forall y)(\exists n)(nx > y) \, .$$

We have $A \subseteq B \Rightarrow A \cap B = A$ and $A \cup B = B$.

Named Mathematical Spaces

$\mathbb{R}$ = the real numbers between $-\infty$ and $+\infty$

$\mathbb{C}$ = the complex numbers

$\mathbb{Z}$ = the integers ($\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$)

$\mathbb{N}$ = the "natural numbers" ($\mathbb{N} = \{0, 1, 2, 3, \cdots\}$)

Other symbols

$\natural$ means "commutes": '$A \natural B$' means $AB = BA$

Mappings

A *mapping* f from one set X to another set Y (possibly the same as X) takes each element $x \in X$ to a unique element y of Y. We write then $y = f(x)$. For example, if $X = Y = \mathbb{R}$, then f might be given by $y = f(x) = x^3$. Sometimes instead of $f(x)$ we write $(f\,x)$ or $f \cdot x$ or just plain fx, depending on context.

The concept of a mapping has a number of names commensurate with its centrality in modern mathematics. The words *mapping*, *function*, *map*, *operator*, *transformation*, and *morphism* are all synonyms.

The notation $f : X \to Y$ expresses that f is a mapping from X to Y. The set X is called the *domain* of f and Y is called its *codomain*.

The *identity mapping* (usually written 1 or, to show its domain, 1_X) takes every element of X to itself: $f(x) = x$.

The *composition* f of two mappings f_1 and f_2 is written $f_1 \circ f_2$ or just $f_1 f_2$, and defined by $f(x) = f_1(f_2(x))$. (For the composition to be meaningful, we need that domain(f_1) = codomain(f_2).) For example, if $f_1(x) = x + 2$ and $f_2(x) = x^2$ and $f = f_1 \circ f_2$ then $f(x) = x^2 + 2$.

A mapping f is *injective* iff no two elements x go to the same y; in symbols: $(\forall x_1 \in X)(\forall x_2 \in X)(x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$. For example, $f(x) = x^3$ is injective from $\mathbb{R}$ to $\mathbb{R}$. A mapping f is *surjective* iff every y comes from *some* x; in symbols: $(\forall y \in Y)(\exists x \in X)(y = f(x))$. For example $f(x) = \sin(x)$ is surjective from $\mathbb{R}$ to $[0, 1]$, but it is not injective.

A mapping f is *bijective* iff it is both injective and surjective. In this case it has an *inverse*, that is, a second mapping f^{-1} such that $ff^{-1} = 1$ and $f^{-1}f = 1$.

Spaces and Isomorphisms

Many different kinds of *spaces* occur in mathematics: sets, groups, vector spaces, manifolds, etc. To each *category* of spaces, certain kinds of mappings are especially suited: arbitrary mappings for general sets, linear mappings for vector spaces, homomorphisms for groups, etc. When we write $f : X \to Y$ we normally imply that f is one of the mappings appropriate to spaces of the type X and Y. For example, if V and W were vector spaces, then $L : V \to W$ would normally be a *linear mapping*.

When $f : X \to Y$ possesses an inverse, we say that X and Y are *isomorphic spaces*, and we write

$$X \simeq Y \ .$$

In case $X = Y$, we also say that f is an *automorphism* of X. Objects (such as numbers) that we can attach to spaces and that are unaltered by isomorphisms are called *invariants*. For example, the *dimension* is a natural number we attach to (finite dimensional) vector spaces. It is invariant, because $V \simeq W \Rightarrow \dim V = \dim W$. In fact, it is a *complete* invariant, because vector spaces with the same dimension (and the same field of scalars $\mathbb{R}$ or $\mathbb{C}$) are necessarily isomorphic.

The concept of isomorphism is crucial in physics because of its role in gravity and in gauge theories. When two spaces are isomorphic, they are "structurally identical" (iso-morphic = same-form from Greek). According to the postulate of "general covariance" isomorphic structures describe the same physics and are in that sense identical. The *invariants* will then be the only physically meaningful quantities or "observables". Underlying this postulate is the idea that the points of spacetime have no separate individuality, but only serve as "carriers" for the relationships that correspond to the true physics.

(For more on the notion of Category, see the book by Robert Geroch, *Mathematical Physics* (U. of Chicago Press, 1985).)