Does Locality Fail at Intermediate Length-Scales?*

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Abstract

If quantum gravity implies a fundamental spatiotemporal discreteness, and if its "laws of motion" are compatible with the Lorentz transformations, then physics cannot remain local. One might expect this nonlocality to be confined to the fundamental discreteness scale, but I will present evidence that it survives at much lower energies, yielding for example a nonlocal equation of motion for a scalar field propagating on an underlying causal set.

Assuming that "quantum spacetime" is fundamentally discrete, how might this discreteness show itself? Some of its potential effects are more evident, others less so. The atomic and molecular structure of ordinary matter influences the propagation of both waves and particles in a material medium. Classically, particles can be deflected by collisions and also retarded in their motion, giving rise in particular to viscosity and Brownian motion. In the case of spatio-temporal discreteness, viscosity is excluded by Lorentz symmetry, but fluctuating deviations from rectilinear motion are still possible. Such "swerves" have been described in [1] and [2]. They depend (for a massive particle) on a single phenomenological parameter, essentially a diffusion constant in velocity space. As far as I know, the corresponding analysis for a quantal particle with mass has not been carried out yet, but for massless quanta such as photons the diffusion equation of [1] can be adapted to say something, and it then describes fluctuations of both energy and polarization (but not of direction), as well as a secular "reddening" (or its opposite). A more complete quantal

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story, however, would require that particles be treated as wave packets, raising the general question of how spatiotemporal discreteness affects the propagation of *waves*. Here, the analogy with a material medium suggests effects such as scattering and extinction, as well as possible nonlinear effects. Further generalization to a "second-quantized field" might have more dramatic, if less obvious, consequences. In connection with cosmology, for example, people have wondered how discreteness would affect the hypothetical inflaton field.

So far, I have been assuming that, although the deep structure of spacetime is discrete, it continues to respect the Lorentz transformations. That this is logically possible is demonstrated [3] by the example of causal set (causet) theory [4]. With approaches such as loop quantum gravity, on the other hand, the status of local Lorentz invariance seems to be controversial. Some people have hypothesized that it would be broken or perhaps "deformed" in such a way that the dispersion relations for light would cease to be those of a massless field. Were this the case, empty space could also resist the passage of particles (a viscosity of the vacuum), since there would now be a state of absolute rest. Moreover, reference [5] has argued convincingly that it would be difficult to avoid $O(1)$ renormalization effects that would lead to different quantum fields possessing different effective light cones. Along these lines, one might end up with altogether more phenomenology than one had bargained for.

As already mentioned, the causal set hypothesis avoids such difficulties, but in order to do so, it has to posit a kinematic randomness, in the sense that a spacetime[⋆] M may properly correspond only to causets C that could have been produced by a *Poisson process* in M. With respect to an approximating spacetime M, the causet thus functions as a kind of "random lattice". Moreover, the infinite volume of the Lorentz group implies that such a "lattice" cannot be home to a local dynamics. Rather the "couplings" or "interactions" that describe physical processes occurring in the causet are — of necessity — radically nonlocal.

To appreciate why this must be, let us refer to the process that will be the subject of much of the rest of this paper: propagation of a scalar field ϕ on a background causet

⋆ In this article, "spacetime" will always mean Lorentzian manifold, in particular a continuum.

C that is well approximated by a Minkowski spacetime $M = \mathbb{M}^d$. To describe such a dynamics, one needs to reproduce within C something like the d'Alembertian operator $\Box$, the Lorentzian counterpart of the Laplacian operator ∇^2 of Euclidean space $\mathbb{E}^3$. Locality in the discrete context, if it meant anything at all, would imply that the action of $\Box$ would be built up in terms of "nearest neighbor couplings" (as in fact ∇^2 can be built up, on either a crystalline or random lattice in $\mathbb{E}^3$). But Lorentz invariance contradicts this sort of locality because it implies that, no matter how one chooses to define nearest neighbor, any given causet element $e \in C$ will possess an immense number of them extending throughout the region of C corresponding to the light cone of e in M. In terms of a Poisson process in M we can express this more precisely by saying that the *probability* of any given element e possessing a limited number of nearest neighbors is vanishingly small. Thus, the other elements to which e must be "coupled" by our box operator will be large in number (in the limit infinite), and in any given frame of reference, the vast majority of them will be remote from e. The resulting "action at a distance" epitomizes the maxim that discreteness plus Lorentz invariance entails nonlocality.

If this reasoning is correct, it implies that physics at the Planck scale must be radically nonlocal. (By Planck scale I just mean the fundamental length- or volume-scale associated with the causet or other discrete substratum.) Were it to be confined to the Planck scale, however, this nonlocality would be of limited phenomenological interest despite its deep significance for the underlying theory. But a little thought indicates that things might not be so simple. On the contrary, it is far from obvious that the kind of nonlocality in question can be confined to any scale, because for any given configuration of the field ϕ, the "local couplings" will be vastly outnumbered by the "nonlocal" ones. How then could the latter conspire to cancel out so that the former could produce a good approximation to $\Box\,\phi$, even for a slowly varying ϕ?

When posed like this, the question looks almost hopeless, but I will try to convince you that there is in fact an answer. What the answer seems to say, though, is that one can reinstate locality only conditionally and to a limited extent. At any finite scale λ, some nonlocality will naturally persist, but the scale λ_0 at which it begins to disappear seems to reflect not only the ultraviolet scale l but also an infrared scale R, which we may identify with the age of the cosmos, and which (in a kind of quantum-gravitational echo of Olber's paradox) seems to be needed in order that locality be recovered at all. On the other hand, (the) spacetime (continuum) as such can make sense almost down to $\lambda = l$. We may thus anticipate that, as we coarse-grain up from l to larger and larger sizes λ, we will reach a stratum of reality in which discontinuity has faded out and spacetime has emerged, but physics continues to be nonlocal. One would expect the best description of this stratum to be some type of nonlocal field theory; and this would be a new sort of manifestation of discreteness: not as a source of fluctuations, but as a source of nonlocal phenomena.

Under still further coarse-graining, this nonlocality should disappear as well, and one might think that one would land for good in the realm of ordinary quantum field theory (and its further coarse-grainings). However, there is reason to believe that locality would fail once again when cosmic dimensions were reached; in fact, the non-zero cosmological constant predicted on the basis of causet theory is very much a nonlocal reflection, on the largest scales, of the underlying discreteness. It is a strictly quantal effect, however, and would be a very different sort of residue of microscopic discreteness than what I'll be discussing here.

These introductory remarks express in a general way most of what I want to convey in this paper, but before getting to the technical underpinnings, let me just (for shortage of space) list some other reasons why people have wanted to give up locality as a fundamental principle of spacetime physics: to cure the divergences of quantum field theory (e.g. [6]); to obtain particle-like excitations of a spin-network or related graph [7]; to give a realistic and deterministic account of quantum mechanics (the Bohmian interpretation is both nonlocal and acausal, for example); to let information escape from inside a black hole (e.g. [8]); to describe the effects of hidden dimensions in "brane world" scenarios; to reduce quantum gravity to a flat-space quantum field theory via the so called AdS-CFT correspondence; to make room for non-commuting spacetime coordinates. (This "non-commutative geometry" reason is perhaps the most suggestive in the present context, because it entails a hierarchy of scales analogous to the scales l, λ_0 and R. On the "fuzzy sphere" in particular, the non-commutativity scale λ_0 is the geometric mean between the effective ultraviolet cutoff l and the sphere's radius R.)

Three D'Alembertians for two-dimensional causets

The scalar field on a causet offers a simple model for the questions we are considering. In the continuum its equations of motion take — at the classical level — the simple form

$\Box\,\phi = 0$, assuming (as we will) that the mass vanishes. In order to make sense of this equation in the causet, we "merely" need to give a meaning to the D'Alembertian operator $\Box$. This is not an easy task, but it seems less difficult than giving meaning to, for example, the gradient of ϕ (which for its accomplishment would demand that we define a concept of vectorfield on a causet). Of course, one wants ultimately to treat the quantum case, but one would expect a definition of $\Box$ to play a basic role there as well, so in seeking such a definition we are preparing equally for the classical and quantal cases.

In the causal set it is natural to take a scalar field to be simply a mapping ϕ of the causet into the real or complex numbers, as the case may be. If we then assume that $\Box$ should act linearly on ϕ (not as obvious as one might think!), our task reduces to the finding of a suitable matrix B_{xy} to play the role of $\Box$, where the indices x, y range over the elements of the causet C. We will also require that B be "retarded" or "causal" in the sense that $B_{xy} = 0$ whenever x is spacelike to, or causally precedes y. In the first place, this is helpful classically, since it allows one to propagate a solution ϕ forward iteratively, element by element (assuming that the diagonal elements B_{xx} do not vanish). It might similarly be advantageous quantally, if the path integration is to be conducted in the Schwinger-Kel'dysh manner.

First approach through the Green function

I argued above that no matrix B that (approximately) respects the Lorentz transformations can reproduce a local expression like the D'Alembertian unless the majority of terms cancel miraculously in the sum, $\sum\limits_y B_{xy}\phi_y =: (B\phi)_x$, that corresponds to $\Box\,\phi(x)$.

Simulations by Alan Daughton [9], continued by Rob Salgado [10], provided the first evidence that the required cancellations can actually be arranged for without appealing to anything other than the intrinsic order-structure of the causet. In this approach one notices that, although in the natural order of things one begins with the D'Alembertian and "inverts" it to obtain its Green function G, the result in $1+1$-dimensions is so simple that the procedure can be reversed. In fact, the *retarded* Green function $G(x,y) = G(x-y)$ in $\mathbb{M}^2$ is (with the sign convention $\Box = -\partial^2/\partial t^2 + \partial^2/\partial x^2$) just the step function with magnitude $-1/2$ and support the future of the origin (the future light cone together with its interior). Moreover, thanks to the conformal invariance of $\Box$ in $\mathbb{M}^2$, the same expression remains valid in the presence of spacetime curvature.

Not only is this continuum expression very simple, but it has an obvious counterpart in the causal set, since it depends on nothing more than the causal relation between the two spacetime points x and y. Letting the symbol $<$ denote (strict) causal precedence in the usual way, we can represent the causet C as a matrix whose elements C_{xy} take the value 1 when $x < y$ and 0 otherwise. The two-dimensional analog G of the retarded Green function is then just $-1/2$ times (the transpose of) this matrix.

From these ingredients, one can concoct some obvious candidates for the matrix B. The one that so far has worked best is obtained by symmetrizing G_{xy} and then inverting it. More precisely, what has been done is the following: begin with a specific region $R \subset \mathbb{M}^2$ (usually chosen to be an order-interval, the diamond-shaped region lying causally between a timelike pair of points); randomly sprinkle N points $x_i,\ i = 1 \dots N$ into R; let C be the causet with these points as substratum and the order-relation $<$ induced from $\mathbb{M}^2$; for any "test" scalar field ϕ on R, let $\phi_i = \phi(x_i)$ be the induced "field" on C; build the $N \times N$ matrix G and then symmetrize and invert to get B, as described above; evaluate $B(\phi, \psi) = \sum_{ij} B_{ij}\phi_i\psi_j$ for ϕ and ψ drawn from a suite of test functions on R; compare with the continuum values, $\int d^2x\, \phi(x) \,\Box\, \psi(y)\, d^2y$.

For test functions that vanish to first order on the boundary ∂R of R, and that vary slowly on the scale set by the sprinkling density, the results so far exhibit full agreement between the discrete and continuum values [9][10]. Better agreement than this, one could not have hoped for in either respect: Concerning boundary terms, the heuristic reasoning that leads one to expect that inverting a Green function will reproduce a discretized version of $\Box$ leaves open its behavior on ∂R. Indeed, one doesn't really know what continuum expression to compare with: If our fields don't vanish on ∂R, should we expect to obtain an approximation to $\int dxdy\phi(x) \,\Box\, \psi(y)$ or $\int dxdy(\nabla\phi(x), \nabla\psi(y))$ or $\dots$? Concerning rapidly varying functions, it goes without saying that, just as a crystal cannot support a sound wave shorter than the interatomic spacing, a causet cannot support a wavelength shorter than l. But unlike with crystals, this statement requires some qualification because the notion of wavelength is frame-dependent. What is red light for one inertial observer is blue light for another. Given that the causet can support the red wave, it must be able to support the blue one as well, assuming Lorentz invariance in a suitable sense. Conversely, such paired fields can be used to test the Lorentz invariance of B. To the limited extent that this important test has been done, the results have also been favorable.

On balance, then, the work done on the Green function approach gives cause for optimism that "miracles do happen". However, the simulations have been limited to the flat case, and, more importantly, they do not suffice (as of yet) to establish that the discrete D'Alembertian B is truly frame independent. The point is that although G itself clearly is Lorentz invariant in this sense, its inverse (or rather the inverse of the symmetrized G) will in general depend on the region R in which one works. Because this region is not itself invariant under boosts, it defines a global frame that could find its way into the resulting matrix B. Short of a better analytic understanding, one is unable to rule out this subtle sort of frame dependence, although the aforementioned limited tests provide evidence against it.

Moreover, the Green function prescription itself is of limited application. In addition to two dimensions, the only other case where a similar prescription is known is that of four dimensions *without* curvature, where one can take for G the "link matrix" instead of the "causal matrix".

Interestingly enough, the potential for Lorentz-breaking by the region R does not arise if one works exclusively with retarded functions, that is, if one forms B from the original retarded matrix G, rather than its symmetrization.[†] Unfortunately, computer tests with the retarded Green function have so far been discouraging on the whole (with some very recent exceptions). Since, for quite different reasons, it would be desirable to find a retarded representation of $\Box$, this suggests that we try something different.

Retarded couplings along causal links

Before taking leave of the Green function scheme just described, we can turn to it for one more bit of insight. If one examines the individual matrix elements B_{xy} for a typical sprinkling, one notices first of all that they seem to be equally distributed among positive and negative values, and second of all that the larger magnitudes among them are concentrated "along the light cone"; that is, B_{xy} tends to be small unless the proper distance between x and y is near zero. The latter observation may remind us of a collection of "nearest neighbor couplings", here taken in the only possible Lorentz invariant sense: that of small

[†] One needs to specify a nonzero diagonal for F.

proper distance. The former observation suggests that a recourse to oscillating signs might be the way to effect the "miraculous cancellations" we are seeking.

The suggestion of oscillating signs is in itself rather vague, but two further observations will lead to a more quantitative idea, as illustrated in figure 1. Let a be some point in $\mathbb{M}^2$, let b and c be points on the right and left halves of its past lightcone (a "cone" in $\mathbb{M}^2$ being just a pair of null rays), and let d be the fourth point needed to complete the rectangle. If (with respect to a given frame) all four points are chosen to make a small square, and if ϕ is slowly varying (in the same frame), then the combination $\phi(a) + \phi(d) - \phi(b) - \phi(c)$ converges, after suitable normalization, to $-\Box\,\phi(a)$ as the square shrinks to zero size. (By Lorentz invariance, the same would have happened even if we had started with a rectangle rather than a square.) On the other hand, four other points obtained from the originals by a large boost will form a long skinny rectangle, in which the points a and b (say) are very close together, as are c and d. Thanks to the profound identity, $\phi(a)+\phi(d)-\phi(b)-\phi(c) = \phi(a)-\phi(b)+\phi(d)-\phi(c)$, we will obtain only a tiny contribution from this rectangle — exactly the sort of cancellation we were seeking! By including not only this rectangle, but all of the boosts of the original square, we might thus hope to do justice to the Lorentz group without bringing in the unwanted contributions we have been worrying about.

Comparison with the D'Alembertian in one dimension leads to a similar idea, which in addition works a bit better in the causet, where elements corresponding to the type of "null rectangles" just discussed don't really exist. In $\mathbb{M}^1$, which is just the real line, $\Box\,\phi$ reduces (up to sign) to $\partial^2\phi/\partial t^2$, for which a well known discretization is $\phi(a)-2\phi(b)+\phi(c)$, a, b and c being three evenly spaced points. Such a configuration *does* find correspondents in the causet, for example 3-chains $x < y < z$ such that no element other than y lies causally between x and z Once again, any single one of these chains (partly) determines a frame, but the collection of all of them does not. Although these examples should not be taken too seriously (compare the sign in equation (1) below), they bring us very close to the following scheme. [♭]

[♭] A very similar idea was suggested once by Steve Carlip

Imagine a causet C consisting of points sprinkled into a region of $\mathbb{M}^2$, and fix an element $x \in C$ at which we would like to know the value of $\Box\,\phi$. We can divide the ancestors of x (those elements that causally precede it) into "layers" according to their "distance from x", as measured by the number of intervening elements. Thus layer 1 comprises those y which are *linked* to x in the sense that $y < x$ with no intervening elements, layer 2 comprises those $y < x$ with only a single element z such that $y < z < x$, etc. (Figure 2 illustrates the definition of the layers.) Our prescription for $\Box\,\phi(x)$ is then to take some combination, with alternating signs, of the first few layers, the specific coefficients to be chosen so that the correct answers are obtained from suitably simple test functions. Perhaps the simplest combination of this sort is

$$B\phi(x) = \frac{4}{l^2}\left(-\frac{1}{2}\phi(x) + \left(\sum_1 -2\sum_2 + \sum_3\right)\phi(y)\right) \tag{1}$$

where the three sums $\sum$ extend over the first three layers as just defined, and l is the fundamental length-scale associated with the sprinkling, normalized so that each sprinkled point occupies, on average, an area of l^2. The prescription (1) yields a candidate for the "discrete D'Alembertian" B which is *retarded*, unlike our earlier candidate based on the symmetrized Green function. In order to express this new B explicitly as a matrix, let $n(x, y)$ denote the cardinality of the order-interval $\langle y, x\rangle = \{z \in C | y < z < x\}$, or in other words the number of elements of C causally between y and x. Then, assuming that $x \geq y$, we have from (1),

$$\frac{l^2}{4}B_{xy} = \begin{cases} -1/2 & \text{for } x = y \\ 1, -2, 1, \text{ according as } n(x,y) \text{ is } 0, 1, 2, \text{ respectively,} & \text{for } x \neq y \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Now let ϕ be a fixed test function of compact support on $\mathbb{M}^2$, and let x (which we will always take to be included in C) be a fixed point of $\mathbb{M}^2$. If we apply B to ϕ we will of course obtain a random answer $B\phi(x)$ depending on the random sprinkling of $\mathbb{M}^2$. However, one can prove that the *mean* of this random variable converges to $\Box\,\phi(x)$ in the continuum limit $l \to 0$:

$$\mathbb{E} \sum_y B_{xy}\phi_y \quad \underset{l\to 0}{\to} \quad \Box\,\phi(x)\ , \tag{3}$$

where $\mathbb{E}$ denotes expectation with respect to the Poisson process that generates the sprinkled causet C. [The proof rests on the following facts. Let us limit the sprinkling to an "interval" (or "causal diamond") $\mathfrak{X}$ with x as its top vertex. For test functions that are polynomials of low degree, one can evaluate the mean in terms of simple integrals over $\mathfrak{X}$ — for example the integral $\int dudv/l^2 \exp\{-uv/l^2\}\ \phi(u,v)$ — and the results agree with $\Box\,\phi(x)$, up to corrections that vanish like powers of l or faster.]

In a sense, then, we have successfully reproduced the D'Alembertian in terms of a causet expression that is *fully intrinsic* and therefore automatically *frame-independent*. Moreover, the matrix B, although it introduces nonlocal couplings, does so only on Planckian scales, which is to say, on scales no greater than demanded by the discreteness itself. *

But is our "discrete D'Alembertian" B really a satisfactory tool for building a field theory on a causet? The potential problem that suggests the opposite conclusion concerns the fluctuations in (1), which grow with N rather than dying away. (This growth is indicated by theoretical estimates and confirmed by numerical simulations.) Whether this problem is fatal or not is hard to say. For example, in propagating a classical solution ϕ forward in time through the causet, it might be that the fluctuations in ϕ induced by those in (1) would remain small when averaged over many Planck lengths, so that the coarse-grained field would not see them. But if this is true, it remains to be demonstrated. And in any case, the fluctuations would be bound to affect even the coarse-grained field when they became big enough. For the remainder of this paper, I will assume that large fluctuations are not acceptable, and that one consequently needs a different B that will yield the desired answer not only on average, but (with high probability) in each given case. For that purpose, we will have to make more complicated the remarkably simple ansatz (2) that we arrived at above.

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* It is not difficult to convince oneself that the limit in (3) sets in when l shrinks below the characteristic length associated with the function ϕ; or vice versa, if we think of l as fixed, $B\phi$ will be a good approximation to $\Box\,\phi$ when the characteristic length-scale λ over which ϕ varies exceeds l: $\lambda \gg l$. But this means in turn that $(B\phi)(x)$ can be sampling ϕ *in effect* only in a neighborhood of x of characteristic size l. Although B is thoroughly nonlocal at a fundamental level, the scale of its effective nonlocality in application to slowly varying test functions is (in the mean) thus no greater than l.

Damping the fluctuations

To that end, let us return to equation (3) and notice that $\mathbb{E}(B\phi) = (\mathbb{E}B)\phi$, where what I have just called $\mathbb{E}B$ is effectively a continuum integral-kernel $\overline{B}$ in $\mathbb{M}^2$. That is to say, when we average over all sprinklings to get $\mathbb{E}B\phi(x)$, the sums in (1) turn into integrals and there results an expression of the form $\int \overline{B}(x-y)\phi(y)d^2y$, where $\overline{B}$ is a retarded, continuous function that can be computed explicitly. Incorporating into $\overline{B}$ the δ-function answering to $\phi(x)$ in (1), we get for our kernel (when $x > y$),

$$\overline{B}(x-y) = \frac{4}{l^4}\, p(\xi)e^{-\xi} \;-\; \frac{2}{l^2}\delta^{(2)}(x-y)\ , \tag{4}$$

where $p(\xi) = 1 - 2\xi + \frac{1}{2}\xi^2$, $\xi = v/l^2$ and $v = \frac{1}{2}||x-y||^2$ is the volume (i.e. area) of the order-interval in $\mathbb{M}^2$ delimited by x and y. The convergence result (3) then states that, for ϕ of compact support,

$$\int \overline{B}(x-y)\phi(y)d^2y \quad \underset{l\to 0}{\to} \quad \Box\,\phi(x) \tag{5}$$

Notice that, as had to happen, $\overline{B}$ is Lorentz-invariant, since it depends only on the invariant interval $||x-y||^2 = |(x-y)\cdot(x-y)|$. †

Observe, now, that the fundamental discreteness-length has all but disappeared from our story. It remains only in the form of a parameter entering into the definition (4) of the integral kernel $\overline{B}$. As things stand, this parameter reflects the scale of microscopic physics from which $\overline{B}$ has emerged (much as the diffusion constants of hydrodynamics reflect atomic dimensions). But nothing in the definition of $\overline{B}$ per se forces us to this identification. If in (4) we replace l by a freely variable length, and if we then follow the

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† The existence of a Lorentz-invariant kernel $\overline{B}(x)$ that yields (approximately) $\Box\,\phi$ might seem paradoxical, because one could take the function ϕ itself to be Lorentz invariant (about the origin $x = 0$, say), and for such a ϕ the integrand in (5) would also be invariant, whence the integral would apparently have to diverge. This divergence is avoided for compactly supported ϕ, of course, because the potential divergence is cut off where the integrand goes to zero. But what is truly remarkable in the face of the counter-argument just given, is that the answer is insensitive to the size of the supporting region. With any reasonable cutoff and reasonably well behaved test functions, the integral still manages to converge to the correct answer as the cutoff is taken to infinity. Nevertheless, this risk of divergence hints at the need we will soon encounter for some sort of infrared cutoff-scale.

Jacobian dictum, "Man muss immer umkehren" [♭] we can arrive at a modification of the discrete D'Alembertian B for which the unwanted fluctuations are damped out by the law of large numbers.

Carrying out the first step, let us replace $1/l^2$ in (4) by a new parameter K. We obtain a new continuum approximation to $\Box$,

$$\overline{B}_K(x-y) = 4K^2\, p(\xi)e^{-\xi} \;-\; 2K\delta^{(2)}(x-y) \;, \tag{6}$$

whose associated nonlocality-scale is not l but the length $K^{-1/2}$, which we can take to be much larger than l. Retracing the steps that led from the discrete matrix (2) to the continuous kernel (4) then brings us to the following causet expression that yields (6) when its sprinkling-average is taken:

$$B_K\phi(x) = \frac{4\varepsilon}{l^2}\left(-\frac{1}{2}\phi(x) + \varepsilon\sum_{y<x} f(n(x,y),\varepsilon)\ \phi(y)\right) \;, \tag{7}$$

where $\varepsilon = l^2 K$, and

$$f(n,\varepsilon) = (1-\varepsilon)^n\left(1 - \frac{2\varepsilon n}{1-\varepsilon} + \frac{\varepsilon^2 n(n-1)}{2(1-\varepsilon)^2}\right) \;. \tag{7a}$$

For $K = 1/l^2$ we recover (1). In the limit where $\varepsilon \to 0$ and $n \to \infty$, $f(n,\varepsilon)$ reduces to the now familiar form $p(\xi)e^{-\xi}$ with $\xi = n\varepsilon$. That is, we obtain in this limit the Montecarlo approximation to the integral $\overline{B}_K\phi$ induced by the sprinkled points. (Conversely, $p(\xi)e^{-\xi}$ can serve as a lazybones' alternative to (7a)).

Computer simulations show that $B_K\phi(x)$ furnishes a good approximation to $\Box\,\phi(x)$ for simple test functions, but this time one finds that the fluctuations *also* go to zero with l, assuming the physical nonlocality scale K remains fixed as l varies. For example, with $N = 2^9$ points sprinkled into the interval in $\mathbb{M}^2$ delimited by $(t,x) = (\pm 1, 0)$, and with the test functions $\phi = 1, t, x, t^2, x^2, tx$, the fluctuations in $B_K\phi(t=1, x=0)$ for $\varepsilon = 1/64$ range from a standard deviation of 0.53 (for $\phi = x^2$) to 1.32 (for $\phi = 1$); and they die out roughly like $N^{-1/2}$ (as one might have expected) when K is held fixed as N increases.

[♭] "One must always reverse direction."

The means are accurate by construction, in the sense that they exactly⋆ reproduce the continuum expression $\overline{B}_K\phi$ (which in turn reproduces $\Box\,\phi$ to an accuracy of around 1% for $K \gtrsim 200$.) (It should also be possible to estimate the fluctuations analytically, but I have not tried to do so.)

In any case, we can conclude that "discretized D'Alembertians" suitable for causal sets do exist, a fairly simple one-parameter family of them being given by (7). The parameter ε in that expression determines the scale of the nonlocality via $\varepsilon = Kl^2$, and it must be $\ll 1$ if we want the fluctuations in $B\phi$ to be small. In other words, we need a significant separation between the two length-scales l and $\lambda_0 = K^{-1/2} = l/\sqrt{\varepsilon}$.

Higher dimensions

So far, we have been concerned primarily with two-dimensional causets (ones that are well approximated by two-dimensional spacetimes). Moreover, the quoted result, (3) cum (6), has been proved only under the additional assumption of flatness, although it seems likely that it could be extended to the curved case. More important, however, is finding D'Alembertian operators/matrices for four and other dimensions. It turns out that one can do this systematically in a way that generalizes what we did in two dimensions.

Let me illustrate the underlying ideas in the case of four dimensional Minkowski space $\mathbb{M}^4$. In $\mathbb{M}^2$ we began with the D'Alembertian matrix B_{xy}, averaged over sprinklings to get $\overline{B}(x-y)$, and "discretized" a rescaled $\overline{B}$ to get the matrix $(B_K)_{xy}$. It turns out that this same procedure works in 4-dimensions if we begin with the coefficient pattern $1\ -3\ 3\ -1$ instead of $1\ -2\ 1$.

To see why it all works, however, it is better to start with the integral kernel and not the matrix (now that we know how to pass between them). In $\mathbb{M}^2$ we found $\overline{B}$ in the form of a delta-function plus a term in $p(\xi)\exp(-\xi)$, where $\xi = Kv(x,y)$, and $v(x,y)$ was the volume of the order-interval $\langle y,\, x\rangle$, or equivalently — in $\mathbb{M}^2$ — Synge's "world function". In other dimensions this equivalence breaks down and we can imagine using either the world function or the volume (one being a simple power of the other, up to a

⋆ Strictly speaking, this assumes that the number of sprinkled points is Poisson distributed, rather than fixed.

multiplicative constant). Whichever one chooses, the real task is to find the polynomial $p(\xi)$ (together with the coefficient of the companion delta-function term).

To that end, notice that the combination $p(\xi)\exp(-\xi)$ can always be expressed as the result of a differential operator $\mathcal{O}$ in $\partial/\partial K$ acting on $\exp(-\xi)$. But then,

$$\int p(\xi)\exp(-\xi)\phi(x)dx = \int \mathcal{O}\exp(-\xi)\phi(x)dx = \mathcal{O}\int \exp(-\xi)\phi(x)dx \equiv \mathcal{O}J\ .$$

We want to choose $\mathcal{O}$ so that this last expression yields the desired results for test functions that are polynomials in the coordinates x^μ of degree two or less. But the integral J has a very simple form for such ϕ. Up to contributions that are negligible for large K, it is just a linear combination of terms of the form $1/K^n$ or $\log K/K^n$. Moreover the only monomials that yield logarithmic terms are (in $\mathbb{M}^2$) $\phi = t^2$, $\phi = x^2$, and $\phi = 1$. In particular the monomials whose D'Alembertian vanishes produce only $1/K$, $1/K^2$ or $O(1/K^3)$, with the exception of $\phi = 1$, which produces a term in $\log K/K$. These are the monomials that we don't want to survive in $\mathcal{O}J$. On the other hand $\phi = t^2$ and $\phi = x^2$ both produce the logarithmic term $\log K/K^2$, and we do want them to survive. Notice further, that the survival of *any* logarithmic terms would be bad, because, for dimensional reasons, they would necessarily bring in an "infrared" dependence on the overall size of the region of integration. Taking all this into consideration, what we need from the operator $\mathcal{O}$ is that it remove the logarithms and annihilate the terms $1/K^n$. Such an operator is

$$\mathcal{O} = \frac{1}{2}(H+1)(H+2) \qquad \text{where} \quad H = K\frac{\partial}{\partial K}$$

is the homogeneity operator. Applying this to $\exp(-\xi)$ turns out to yield precisely the polynomial $p(\xi)$ that we were led to above in another manner, explaining in a sense why this particular polynomial arises. (The relation to the binomial coefficients, traces back to an identity, proved by Joe Henson, that expresses $(H+1)(H+2)...(H+n)\exp(-K)$ in terms of binomial coefficients.) Notice finally that $(H+1)(H+2)$ does *not* annihilate $\log K/K$; but it converts it into $1/K$, which can be canceled by adding a delta-function to the integral kernel, as in fact we did. (It could also have been removed by a further factor of $(H+1)$.)

The situation for $\mathbb{M}^4$ is very similar to that for $\mathbb{M}^2$. The low degree monomials again produce terms in $1/K^n$ or $\log K/K^n$, but everything has an extra factor of $1/K$. Therefore

$\mathcal{O} = \frac{1}{6}(H+1)(H+2)(H+3)$ is a natural choice and leads to a polynomial based on the binomial coefficients of $(1-1)^3$ instead of $(1-1)^2$. From it we can derive both a causet D'Alembertian and a nonlocal, retarded deformation of the continuum D'Alembertian, as before. It remains to be confirmed, however, that these expressions enjoy all the advantages of the two-dimensional operators discussed above. It also remains to be confirmed that these advantages persist in the presence of curvature (but not, of course, curvature large compared to the nonlocality scale K that one has introduced).

It seems likely that the same procedure would yield candidates for retarded D'Alembertians in all other spacetime dimensions.

Continuous nonlocality, Fourier transforms and Stability

In the course of the above reflexions, we have encountered some D'Alembertian matrices for the causet and we have seen that the most promising among them contain a free parameter K representing an effective nonlocality scale or "meso-scale", as I will sometimes call it. For processes occurring on this scale (assuming it is much larger than the ultraviolet scale l so that a continuum approximation makes sense) one would expect to recognize an effective nonlocal theory corresponding to the retarded two-point function $\overline{B}_K(x,y)$. For clarity of notation, I will call the corresponding operator on scalar fields $\underset{K}{\Box}$, rather than $\overline{B}_K$.

Although its nonlocality stems from the discreteness of the underlying causet, $\underset{K}{\Box}$ is a perfectly well defined operator in the continuum, which can be studied for its own sake. At the same time, it can help shed light on some questions that arise naturally in relation to its causet cousin B_{xy}.

One such question (put to me by Ted Jacobson) asks whether the evolution defined in the causet by B_{xy} is stable or not. This seems difficult to address as such except by computer simulations, but if we transpose it to a continuum question about $\underset{K}{\Box}$, we can come near to a full answer. Normally, one expects that if there were an instability then $\underset{K}{\Box}$ would possess an "unstable mode" (quasinormal mode), that is, a spacetime function ϕ of the form $\phi(x) = \exp(ik\cdot x)$ satisfying $\underset{K}{\Box}\,\phi = 0$, with the imaginary part of the wave-vector k being future-timelike. †

Now by Lorentz invariance, $\underset{K}{\Box}\,\phi$ must be expressible in terms of $z = k \cdot k$, and it is not too difficult to reduce it to an "Exponential integral" Ei in z. This being done, some exploration in Maple strongly suggests that the only solution of $\underset{K}{\Box}\,\exp(ik\cdot x) = 0$ is $z = 0$, which would mean the dispersion relation was unchanged from the usual one, $\omega^2 = k^2$. If this is so, then no instabilities can result from the introduction of our nonlocality scale K, since the solutions of $\underset{K}{\Box}\,\phi = 0$ are precisely those belonging to the usual D'Alembertian. The distinction between propagation based on the latter and propagation based on $\underset{K}{\Box}$ would repose only on the different relationship that ϕ would bear to its sources; propagation in empty (and flat) space would show no differences at all. (The massive case might tell a different story, though.)

Fourier transform methods more generally

What we've just said is essentially that the Fourier transform of $\underset{K}{\Box}$ vanishes nowhere in the complex z-plane ($z \equiv k \cdot k$), except at the origin. But this draws our attention to the Fourier transform as yet another route for arriving at a nonlocal D'Alembertian. Indeed, most people investigating deformations of $\Box$ seem to have thought of them in this way, including for example [6]. They have written down expressions like $\Box\exp(\Box/K)$, but without seeming to pay much attention to whether such an expression makes sense in a spacetime whose signature has not been Wick rotated to $(++++)$. In contrast, the operator $\underset{K}{\Box}$ of this paper was defined directly in "position space" as an integral kernel, not as a formal function of $\Box$. Moreover, because it is retarded, its Fourier transform is rather special .... By continuing in this vein, one can come up with a third derivation of $\underset{K}{\Box}$ as (apparently) the simplest operator whose Fourier transform obeys the analyticity and boundedness conditions required in order that $\underset{K}{\Box}$ be well-defined and retarded.

† One might question whether $\underset{K}{\Box}\,\phi$ is defined at all for a general mode since the integral that enters into its definition might diverge, but for a putative unstable mode, this should not be a problem because the integral has its support precisely where the mode dies out: toward the past.

The Fourier transform itself can be given in many forms, but the following is among the simplest:

$$\underset{K}{\Box} e^{ik\cdot x}|_{x=0} = \frac{2z}{i} \int_0^\infty dt \, \frac{e^{itz/K}}{(t-i)^2} \tag{8}$$

where here, $z = -k \cdot k/2$.

It would be interesting to learn what operator would result if one imposed "Feynman boundary conditions" on the inverse Fourier transform of this function, instead of "causal" ones.

What next?

Equations (7) and (6) offer us two distinct, but closely related, versions of $\Box$, one suited to a causet and the other being an effective continuum operator arising as an average or limit of the first. Both are retarded and each is Lorentz invariant in the relevant sense. How can we use them? First of all, we can take up the questions about wave-propagation raised in the introduction, looking in particular for deviations from the simplified model of [11] based on "direct transmission" from source to sink (a model that has much in common with the approach discussed above under the heading "Approach through the Green function"). Equation (7) in particular, would let us propagate a wave-packet through the causet and look for some of the effects indicated in the introduction, like "swerves", scattering and extinction. These of course hark back directly to the granularity of the causet, but even in the continuum limit the nonlocality associated with (6) might modify the field emitted by a given source in an interesting manner; and this would be relatively easy to analyze.

Also relatively easy to study would be the effect of the nonlocality on free propagation in a curved background. Here one *would* expect some change to the propagation law. Because of the retarded character of $\underset{K}{\Box}$, one might also expect to see some sort of induced CPT violation in an expanding cosmos. Because (in a quantal context) this would disrupt the equality between the masses of particles and antiparticles, it would be a potential source of baryon-anti-baryon asymmetry not resting on any departure from thermal equilibrium.

When discreteness combines with spacetime curvature, new issues arise. Thus, propagation of wave-packets in an expanding universe and in a black hole background both raise puzzles having to do with the extreme red shifts that occur in both situations (so-called transplanckian puzzles). In the black hole context, the red shifts are of course responsible for Hawking radiation, but their analysis in the continuum seems to assign a role to modes of exponentially high frequency that arguably should be eschewed if one posits a minimum length. Equation (7) offers a framework in which such questions can be addressed without infringing on Lorentz invariance. The same holds for questions about what happens to wave-packets in (say) a de Sitter spacetime when they are traced backward toward the past far enough so that their frequency (with respect to some cosmic rest frame) exceeds Planckian values. Of course, such questions will not be resolved fully on the basis of classical equations of motion. Rather one will have to formulate quantum field theory on a causet, or possibly one will have to go all the way to a quantal field on a quantal causet (i.e. to quantum gravity). Nevertheless, a better understanding of the classical case is likely to be relevant.

I will not try to discuss here how to do quantum field theory on a causet, or even in Minkowski spacetime with a nonlocal D'Alembertian. That would raise a whole set of new issues, path-integral vs. operator methods and the roles of unitarity and causality being just some of them.[♭] But it does seem in harmony with the aim of this paper to comment briefly on the role of nonlocality in this connection. As we have seen, the ansatz (6) embodies a nonlocal interaction that has survived in the continuum limit, and thus might be made the basis of a nonlocal field theory of the sort that people have long been speculating about.

What is especially interesting from this point of view is the potential for a new approach to renormalization theory (say in flat spacetime $\mathbb{M}^d$). People have sometimes hoped that nonlocality would eliminate the divergences of quantum field theory, but as far as I can see, the opposite is true, at least for the specific sort of nonlocality embodied in (6). In saying this, I'm assuming that the divergences can all be traced to divergences of the Green function $G(x-y)$ in the coincidence limit $x=y$. If this is correct then one would need to soften the high frequency behavior of G, in order to eliminate them. But a glance at (8) reveals that $\underset{K}{\Box}$ has a milder ultraviolet behavior than $\Box$, since its Fourier transform

[♭] I will however echo a comment made earlier: I suspect that one should not try to formulate a path-integral propagator as such; rather one will work with Schwinger-Kel'dysh paths.

goes to a constant at $z = \infty$, rather than blowing up linearly. Correspondingly, one would expect its Green function to be more singular than that of the local operator $\Box$, making the divergences worse, not better. If so, then one must look to the discreteness itself to cure the divergences; its associated nonlocality will not do the job.

But if nonlocality alone cannot remove the need for renormalization altogether, it might nevertheless open up a new and more symmetrical way to arrive at finite answers. The point is that (8) behaves at $z = \infty$ like $1+O(1/z)$, an expression whose reciprocal has exactly the same behavior! The resulting Green function should therefore also be the sum of a delta-function with a regular * function (and the same reasoning would apply in four dimensions). The resulting Feynman diagrams would be finite *except for* contributions from the delta-functions. But these could be removed by hand ("renormalized away"). If this idea worked out, it could provide a new approach to renormalization based on a new type of Lorentz invariant regularization. (Notice that this all makes sense in real space, without the need for Wick rotation.)

How big is λ_0?

From a phenomenological perspective, the most burning question is one that I cannot really answer here: Assuming there are nonlocal effects of the sort considered in the preceding lines, on what length-scales would they be expected to show up? In other words, what is the value of $\lambda_0 = K^{-1/2}$? Although I don't know how to answer this question theoretically, † it is possible to deduce bounds on λ_0 if we assume that the fluctuations in individual values of $\Box\,\phi(causet) = B_K\phi$ are small, as discussed above. Whether such an assumption will

* At worst, it might diverge logarithmically on the light cone, but in that case, the residual divergence could be removed by adjusting the Fourier transform to behave like $1+O(1/z^2)$.

† The question of why l would be so much smaller than λ_0 would join the other "large number" (or "hierarchy") puzzles of physics, like the small size of the cosmological constant Λ. Perhaps the ratio λ_0/l would be set dynamically, say "historically" as a concomitant of the large age and diameter of the cosmos (cf. [12]). If a dynamical mechanism doing this could be discovered, it might also help to explain the current magnitude of Λ, either by complementing the mechanism of [13] with a reason why the value about which Lambda fluctuates is so close to zero, or by offering an alternative explanation altogether.

still seem necessary at the end of the day is of course very much an open question. Not only could a sum over individual elements of the causet counteract the fluctuations (as already mentioned), but the same thing could result from the sum over causets implicit in quantum gravity. This would be a sum of exponentially more terms, and as such it could potentially remove the need for any intermediate nonlocality-scale altogether.

In any case, if we do demand that the fluctuations be elementwise small, then λ_0 is bounded from below by this requirement. (It is of course bounded above by the fact that — presumably — we have not seen it yet.) Although this bound is not easy to analyze, a very crude estimate that I will not reproduce here suggests that we make a small fractional error in $\Box\,\phi$ when (in dimension four)

$$\lambda^2 l^2 R \ll \lambda_0^5 \ ,$$

where λ is the characteristic length-scale associated with the scalar field. On the other hand, even the limiting continuum expression $\underset{K}{\Box}\,\phi$ will be a bad approximation unless $\lambda \gg \lambda_0$. Combining these inequalities yields $\lambda^2 l^2 R \ll \lambda_0^5 \ll \lambda^5$, or $l^2 R \ll \lambda^3$. For smaller λ, accurate approximation to $\Box\,\phi$ is incompatible with small fluctuations. Inserting for l the Planck length[♭] of $10^{-32} cm$ and for R the Hubble radius, yields $\lambda \sim 10^{-12} cm$ as the smallest wavelength that would be immune to the nonlocality. That this is not an extremely small length, poses the question whether observations already exist that could rule out nonlocality on this scale.[⋆]

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♭ This could be an underestimate if a significant amount of coarse-graining of the causet were required for spacetime to emerge.

⋆ Compare the interesting observations (concerning "swerves") in [14]

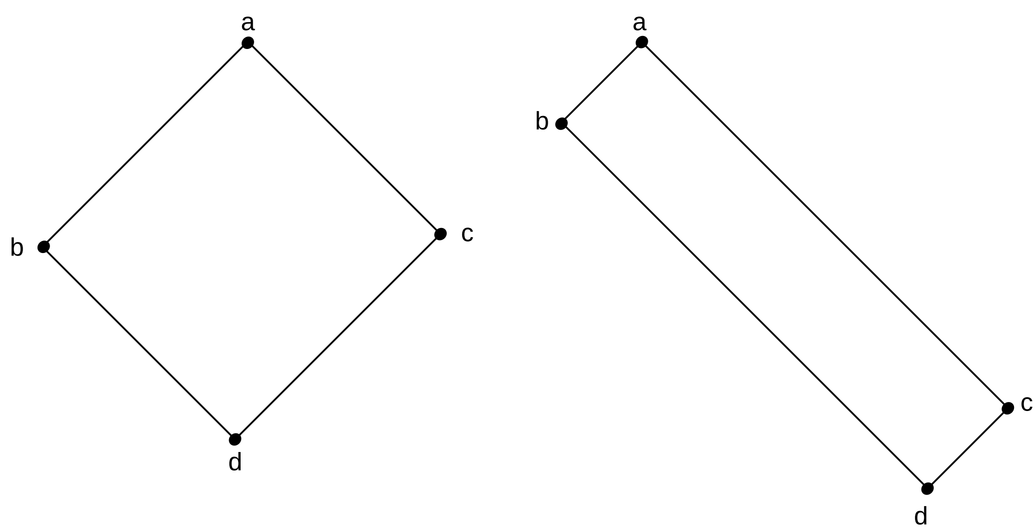

Figure 1. How a miracle might happen

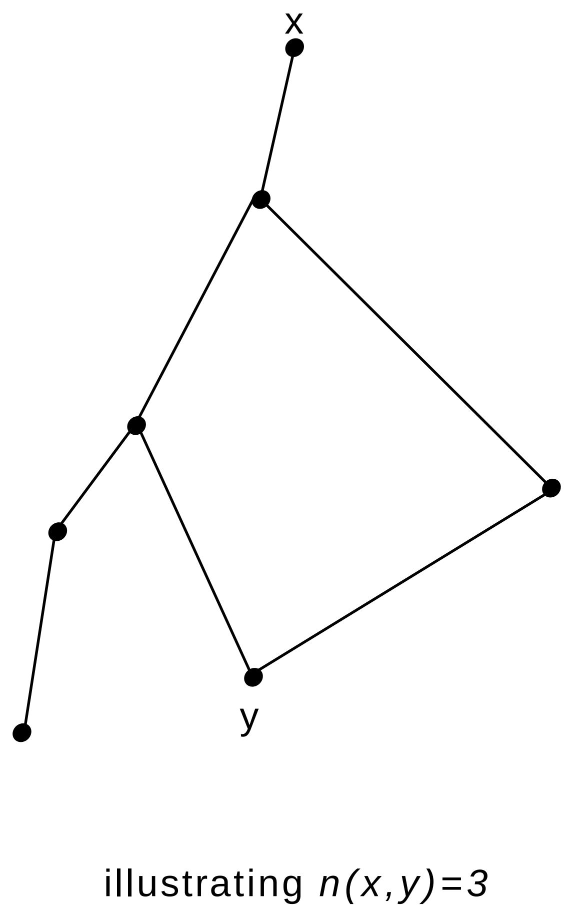

Figure 2. Illustration of the definition of the "proximity measure" $n(a, b)$. For the causet shown in the figure, $n(x, y) = 3$ because three elements intervene causally between y and x. The first layer below element x contains a single element, while the second, third and fourth, contain 2, 1, and 2, elements respectively.

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