

# An exercise in “anhomomorphic logic”<sup>\*</sup>

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## Abstract

A classical logic exhibits a threefold inner structure comprising an algebra of propositions  $\mathfrak{A}$ , a space of “truth values”  $V$ , and a distinguished family of mappings  $\phi$  from propositions to truth values. Classically  $\mathfrak{A}$  is a Boolean algebra,  $V = \mathbb{Z}_2$ , and the admissible maps  $\phi : \mathfrak{A} \rightarrow \mathbb{Z}_2$  are *homomorphisms*. If one admits a larger set of maps, one obtains an anhomomorphic logic that seems better suited to quantal reality (and the needs of quantum gravity). I explain these ideas and illustrate them with three simple examples.

From a certain point of view, the phrase “classical logic” should be used in the plural, not the singular, because the things with which logic deals depend on the “domain of discourse”, and this can vary both with time and with the “system” one has in mind. To each such domain corresponds its own *Boolean algebra*, namely the algebra  $\mathfrak{A}$  of all “questions” one may ask about the system.<sup>1</sup> But there is more to a logic than just a domain of questions together with rules for combining them via *and*, *xor*, *not*, etc. In

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<sup>1</sup> Instead of “question” one also says “predicate”, “proposition” or “event”. I will use these terms interchangeably, and will sometimes refer to  $\mathfrak{A}$  as the “event algebra”, for lack of a better term.

addition, one has the space of “answers”<sup>2</sup> that a question may have, which classically is  $\mathbb{Z}_2 = \{0, 1\} = \{false, true\}$ . And one has also the space of allowed “answering maps”  $\phi : \mathfrak{A} \rightarrow \mathbb{Z}_2$ , which classically coincides with the space of *homomorphisms* from the algebra  $\mathfrak{A}$  to the algebra  $\mathbb{Z}_2$ . To provide such a  $\phi$  (which I will refer to as a *co-event* since it takes events to scalars) is to answer every conceivable question in  $\mathfrak{A}$ , and consequently to give as full a description of the corresponding reality as is classically possible.

Seen in this way, logic has a threefold character, and one might think to generalize the classical setup by altering any one of its basic ingredients: the algebra of propositions  $\mathfrak{A}$ , the space of truth-values  $\mathbb{Z}_2$ , or the possible maps  $\phi$ . I have proposed elsewhere [1] that the paradoxical features of the quantum world can be understood if one fastens on the last of these three possibilities, modifying the nature of the co-events without disturbing either the space of truth-values or the Boolean character of the event algebra  $\mathfrak{A}$ . The change wrought by such a modification can be thought of in different ways. One might say that reality remains what it was for classical logic (namely an individual “history”, in the sense of a collection of particle worldlines, for example<sup>3</sup>), but one accepts that it will sometimes manifest contradictory attributes; or one might say (less provocatively?) that the conception of reality is no longer that of an individual history, but rather of a “non-classical” co-event  $\phi : \mathfrak{A} \rightarrow \mathbb{Z}_2$ .

By terming  $\phi$  “non-classical”, I merely mean that it is no longer required to be a homomorphism. According to classical logic, the truth or falsity of a compound proposition  $P$  like ‘ $A$  or  $B$ ’ is unambiguously determined by the truth or falsity of its individual constituents. In algebraic language, if we know  $\phi(A_i)$  for all the constituents  $A_i$  then we also know  $\phi(P)$ , and the rules for deriving  $\phi(P)$  from the  $\phi(A_i)$  say precisely that  $\phi$  is a homomorphism of  $\mathfrak{A}$  into  $\mathbb{Z}_2$ . In the type of generalization proposed in [1], this is no longer the case. One may therefore call such a generalized logic “anhomomorphic”. (The name “quantal logic” would also be suitable were it not already in use for a different sort of structural modification that replaces  $\mathfrak{A}$  by the collection of subspaces of a Hilbert

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<sup>2</sup> also called “truth values”.

<sup>3</sup> I am emphatically adopting a “histories standpoint”, according to which reality has a “spacetime” nature, not a “spatial” one.

space, which is a lattice but not an algebra at all in the strict sense.<sup>4</sup>) By relaxing the demand that a possible coevent be a homomorphism, one can accommodate contradictions such as that of the Kochen-Specker gedankenexperiment, but of course one needs to limit the coevents in some other manner in order to arrive at a meaningful logical framework. Otherwise one would be unable to infer the truth or falsity of any statement about the world from that of any others, and theoretical predictions would become impossible.

What condition, then, should replace the requirement that co-events be homomorphisms? But first, why do we need to replace it at all? The reason, as explained in [1], is that we are assuming that all dynamical truths can be deduced from the *preclusion* of certain events  $A \in \mathfrak{A}$ , expressed symbolically as  $\phi(A) = 0$ . Among these events are all those for which  $\mu(A) = 0$ , where  $\mu$  is the “quantal measure” furnished by the path integral (see [1]). For example in a diffraction experiment with silver atoms, the event of the atom going to a dark part of the interference pattern is precluded. But in combination with the classical laws of inference this preclusion rule is too powerful. It ends up denying events that clearly do happen. In order to accept all the preclusions furnished by quantum theory, we must therefore give up some of the laws of classical logic. But in doing so, it is natural to be guided by the idea that  $\phi$  should remain as close to being a homomorphism as it can.

The conditions explored briefly in [1] arose in this manner. They dealt well with several of the simplest paradoxes, and they had the virtue of reproducing classical logic when the pattern of preclusions was classical, but it appears that they are still too restrictive (specifically in connection with 4-slit diffraction [2]). In this paper, I will propose a modified set of rules, motivated by the same underlying ideas, and then illustrate the resulting logical scheme by working out some simple examples. In fact I will describe two different schemes, both of which seem so far to be viable.

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<sup>4</sup> One might ask with what “quantum logic” replaces the other two logical ingredients, the space of truth-values and the homomorphism  $\phi$ . As far as I know, this question has never been answered, and therefore no definite picture of reality has been given. Instead, one has focused on maps from the lattice to the unit interval that generalize the classical idea of *probability*, rather than that of truth.

In proposing them, I am not trying to claim that either of the new schemes will prove to be the last word. Rather, I feel that further experimentation with the rules will have to precede any definitive formulation. What I hope will prove to be lasting is the insight that an “objective” interpretation of quantum mechanics can be founded on the concept of preclusion, provided that one generalizes one’s conception of reality by admitting anhomomorphic co-events into one’s logic.

Before turning to the specific schemes, let us look at the question in a somewhat different way. The problem we are faced with can be seen as a clash between two “tendencies of nature”. On the one hand, events  $E$  of very small measure  $\mu(E)$  tend not to happen. (This is one way to think about probability as a “propensity”, in the classical case.) On the other hand, nature tends to avoid “contradictions” among the events, which we can express mathematically by saying that nature favors coevents  $\phi : \mathfrak{A} \rightarrow \mathbb{Z}_2$  that preserve the logical operations *and*, *not*, etc.<sup>5</sup> (This could seem an aesthetic requirement, since a map between two algebras is “most naturally a homomorphism”.) Without seeking a deeper understanding of these two tendencies, we can just accept them and ask what types of possible realities (as co-events) they point to, given that the two tendencies oppose each other to some extent.

If the above viewpoint is valid, then the coevent  $\phi$  that actually occurs (i.e. the coevent that describes what actually happens) results in part from a balancing of two opposite tendencies. How this balancing takes place is something we know only in part. But to the extent that we can guess the full scheme, we can determine which coevents are possible and which are not (or are at least “almost forbidden”). To do so would be to complete the path-integral formalism (which defines  $\mu$ ) by producing a predictive dynamical scheme free of reference to “external observers”. (One might hope to go still farther by formulating the dynamical laws directly in terms of the coevent itself, without referring to  $\mu$  at all; but no road toward that goal is discernible at present.)

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<sup>5</sup> One could also express this by saying that nature tends to observe the “laws of inference” of classical logic. In the type of scheme I am proposing, there will in general be no universal laws of inference, but only concrete inferential relationships that depend on which system (and “initial conditions”) one is dealing with.

To help us in our “balancing act”, we have several guides. The recovery of classical logic in the classical limit (and ultimately of probability) is one such guide. Another is the need to deal adequately with “product systems”, and still others are the need to furnish a realistic account of what happens in a measurement and to respect “relativistic causality”. However, the best single source of inspiration may be the “quantal antinomies”, by which I mean the paradoxical experiments and thought experiments that illustrate why it is so difficult to come up with a satisfying interpretation of the quantal formalism. These include the EPRB experiment, where probabilistic correlations play a role, but there are other examples in which logic alone leads to the paradox, and the latter offer easier starting points for exploring a scheme of “anhomomorphic logic” such as I am proposing.

Before I propose any specific scheme, it would be a good idea to pause to discuss the mathematical structures of  $\mathfrak{A}$ , with respect to which the notion of homomorphism acquires a meaning. We will also need to refer to some of the mathematical structures possessed by the space of coevents, i.e. by the space of functions from  $\mathfrak{A}$  to  $\mathbb{Z}_2$ . (I will call this space  $\mathfrak{A}^*$  for lack of a better symbol.) In the background of both the older and newer schemes is a space  $\Omega$  of “histories” and a *quantal measure*  $\mu$  or *decoherence functional*  $D$  on that space. For brevity, I will not describe any of this in further detail, referring the reader to [1] (and references therein) for more explanation. I will call  $\Omega$  the *sample space*, and its elements “formal trajectories” (by analogy with the case where  $\gamma \in \Omega$  is a set of particle worldlines). And for simplicity I will always take  $\Omega$  to be finite.

On an “extensional” view, the event algebra  $\mathfrak{A}$  is simply (given that  $\Omega$  is finite) the collection of all subsets of  $\Omega$ . (So an element  $A \in \mathfrak{A}$  is just a subset of  $\Omega$ ; “intensionally” it is the corresponding “predicate” or potential “property of reality”.) As such it supports a multitude of operations that make it, among other things, a poset, a distributive lattice with complement, a ring, and an algebra over the finite field  $\mathbb{Z}_2$ . It is a poset<sup>6</sup> in an obvious way, with respect to the ordering given by set inclusion. It is a lattice with respect to the operations of intersection and union. (Indeed a lattice is a special case of a poset.) And it becomes a ring if one interprets the product ‘ $AB$ ’ as ‘ $A \cap B$ ’ and the sum ‘ $A + B$ ’ as the symmetric difference ‘ $A \setminus B \cup B \setminus A$ ’. (Logically, ‘ $AB$ ’ is ‘ $A$  and  $B$ ’, while ‘ $A + B$ ’ is ‘ $A$  xor  $B$ ’ [xor being “exclusive or”].) The fact that  $A + A = 0$  makes  $\mathfrak{A}$  not only a ring but an algebra

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<sup>6</sup> partially ordered set

over the finite field  $\mathbb{Z}_2$ , and the fact that  $AA = A$  makes it a Boolean algebra. Notice in particular that with these definitions,  $\mathfrak{A}$  is a vector space over  $\mathbb{Z}_2$ , which is something that could not have been deduced merely from the fact that it is a lattice with respect to “and” and “or”. This is also a good place to point out that the space  $\mathfrak{A}^*$  of co-events  $\phi$  is also an algebra (in fact a Boolean algebra once again), simply because it is a function-space, and one can define addition and multiplication pointwise:  $(\phi_1\phi_2)(A) = \phi_1(A)\phi_2(A)$  and  $(\phi_1 + \phi_2)(A) = \phi_1(A) + \phi_2(A)$ .

The variety of ways in which we can conceive of  $\mathfrak{A}$  engenders a corresponding variety of notions of what it means for a function  $\phi : \mathfrak{A} \rightarrow \mathbb{Z}_2$  to be a homomorphism. All these definitions agree when  $\phi$  actually is a homomorphism, but when it deviates from being so, they can give rise to different ways to judge by how much it has deviated. This is one source of ambiguity in how best to balance “preclusivity” against “homomorphicity”.

I will call a coevent  $\phi$  *preclusive* if it maps every precluded event  $A$  to zero (= false), and “homomorphic” if it preserves the logical operations, or equivalently if it preserves algebraic sum and product (and  $\phi(1) \equiv \phi(\Omega) = 1$ ). As explained in [1], it cannot in general do both at once. In the schemes to be discussed,  $\phi$  will be strictly preclusive,<sup>7</sup> so it will of necessity be anhomomorphic. The question then becomes in what sense  $\phi$  can remain “close” to a homomorphism without being literally so.

In the scheme proposed in [1] (call it the “linear scheme”), the algebraic aspect of  $\mathfrak{A}$ ’s structure was taken as primary, meaning that preservation of sum (linearity) and of product (“multiplicativity”) were the criteria for it to be homomorphic. Linearity was retained exactly, but multiplicativity was dropped. In place of the latter,  $\phi$  was to be unital<sup>8</sup> and “*minimal*”. And of course it was to be *preclusive*, as we are assuming always. The word minimal here refers to the support of  $\phi$ , which I will define in a moment. One says that  $\phi$  is minimal if there is no (preclusive, linear)  $\phi$  with smaller support. For linear

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<sup>7</sup> It seems harder to relax preclusivity in a controlled way than homomorphicity.

<sup>8</sup> That  $\phi$  is unital says merely that  $\phi(1) = 1$ . For a homomorphism this is automatic except for the trivial  $\phi$  that vanishes identically:  $\phi = 0$ . With events  $A \in \mathfrak{A}$  construed as questions, the event  $1 = \Omega \in \mathfrak{A}$  asks “Does anything at all happen?”, and  $\phi$  is unital iff it answers “Yes” to this question.

$\phi$ ,  $\phi$  is homomorphic iff its support is a single element of  $\Omega$ . Thus one may claim that the smaller its support, the more nearly homomorphic  $\phi$  is. (The minimal such  $\phi$  also *generate* — i.e. they span — the whole vector space of linear preclusive coevents.)

In order to define support, we need some more notation. For  $\gamma \in \Omega$  a formal trajectory, let  $\gamma^* : \mathfrak{A} \rightarrow \mathbb{Z}_2$  be the “containment” map defined by  $\gamma^*(A) = 1$  if  $\gamma \in A$ ,  $\gamma^*(A) = 0$  if  $\gamma \notin A$ . (Instead of thinking of  $\gamma$  as an element of  $\Omega$  one can think of it as an element of  $\mathfrak{A}$  by identifying it with the singleton set  $\{\gamma\}$ . As such it is what is called an *atom* of  $\mathfrak{A}$ , meaning a minimal element when  $\mathfrak{A}$  is regarded as a lattice. Algebraically,  $x$  is an atom if for all  $y$ ,  $xy$  is either 0 or  $x$  itself. In general it simplifies the notation to identify  $\gamma$  with the corresponding atom  $\{\gamma\}$ , and I will normally do so in the sequel.) It is easy to verify that  $\gamma^*$  is a unital homomorphism of  $\mathfrak{A}$  onto  $\mathbb{Z}_2$ . (It thus may be called a “classical coevent”.) In particular it is linear. Moreover the  $\gamma^*$  are a basis for the space  $\mathcal{L}(\mathfrak{A}, \mathbb{Z}_2)$  of linear transformations from  $\mathfrak{A}$  into  $\mathbb{Z}_2$ . Thus every  $\phi \in \mathcal{L}(\mathfrak{A}, \mathbb{Z}_2)$  decomposes uniquely as a sum of the form

$$\phi = \sum_{\gamma \in S} \gamma^* ,$$

where  $S$  is some subset of  $\Omega$  that one may refer to as the *support* of  $\phi$ ,  $\text{supp}(\phi)$ . More generally, any mapping  $\phi$  of  $\mathfrak{A}$  into  $\mathbb{Z}_2$  whatsoever can be expressed as a polynomial in the classical coevents  $\gamma^*$  for  $\gamma \in \Omega$ .<sup>9</sup> One may thus extend the notion of support by defining  $\text{supp}(\phi)$  to be the set of all  $\gamma^*$  occurring in the polynomial.

The two schemes we will explore below are related in opposite ways to the linear scheme. In both of them reality can be identified with a single<sup>10</sup> coevent  $\phi \in \mathfrak{A}^*$ . As stated

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<sup>9</sup> This leads to a useful graphical notation in which, for  $x, y, z \in \Omega$ ,  $x^*$  is represented as a vertex or 0-simplex,  $x^*y^*$  is an edge or 1-simplex,  $x^*y^*z^*$  is a 2-simplex, etc. For  $\phi$  represented in this way and  $A \subseteq \Omega$ ,  $\phi(A) = 1$  iff  $A$  contains an odd number of such simplices.

<sup>10</sup> Here I’m imagining that  $\mathfrak{A}$  includes all possible predicates, including ones pertaining to happenings arbitrarily far in the future (or past). If this is not the case, then one will need to consider more than just a single event-algebra  $\mathfrak{A}$ . There will then be a coevent for each  $\mathfrak{A}$  together with coherence conditions among these “partial coevents”. Furthermore, a coevent which is “minimal” with respect to one such  $\mathfrak{A}$  can fail to remain so when restricted to a subalgebra. This can lead to a kind of “premonition” phenomenon that shows up, for example, in the Hardy gedankenexperiment discussed in [1].

already,  $\phi$  will be preclusive in both schemes ( $\mu(A) = 0 \Rightarrow \phi(A) = 0$ ), but anhomomorphic. The first of the two schemes will be in a sense complementary to the linear scheme. Rather than preserving sum it will preserve product (so I'll call it the "multiplicative scheme"). The second scheme will be a kind of augmented linear scheme, but it will preserve strictly neither sum nor product. For reasons that will become clear, I'll call it the "ideal-based scheme". Since the multiplicative scheme is easier to define, let's begin with it.

In this scheme,  $\phi$  preserves the product,  $AB$ , which in logical terms means that it preserves conjunction, 'and'. Let us also exclude the trivial multiplicative coevents  $\phi = 1$  and  $\phi = 0$ . The axioms for the multiplicative scheme will then include:

- (i)  $\phi \neq 0, 1$
- (ii)  $(\forall A, B \in \mathfrak{A}) \phi(AB) = \phi(A)\phi(B)$

It is relatively easy to work out the general form of such a  $\phi$ , by asking which events are true according to  $\phi$ , i.e. for which  $A \in \mathfrak{A}$  one has  $\phi(A) = 1$ . Suppose  $A \in \mathfrak{A}$ ,  $\phi(A) = 1$  and  $A \subseteq B$ . Then, since  $AB = A$ , we have by multiplicativity,  $\phi(A)\phi(B) = \phi(A) = 1 \Rightarrow \phi(B) = 1$ . Thus  $\phi$  is monotone: any superset of a true event is also true. Now suppose that  $A$  and  $B$  are both true with respect to  $\phi$ . It follows immediately that they have in common some true "subevent" (i.e. subset)  $C \subseteq A, B$ ; for  $A \cap B = AB$  is such a subset and  $\phi(AB) = \phi(A)\phi(B) = 1 \times 1 = 1$ . By induction, there must be a smallest true event  $F \in \mathfrak{A}$  with the property that the other true events coincide with its supersets:<sup>11</sup>

$$\phi(A) = 1 \iff F \subseteq A \tag{1}$$

In this scheme, then, a coevent can be construed as a subset  $F$  of  $\Omega$ , and the rule (1) tells us that, with respect to a given coevent  $\phi$ , reality "has the property  $A$ " iff all the atoms in  $F$  "share this property". Since (1) is a simple generalization of the above definition of  $\gamma^*$ , it is natural to write it as:  $\phi = F^*$ . One can also see that, for any  $F \in \mathfrak{A}$ ,

$$F^* = \prod_{\gamma \in F} \gamma^*$$

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<sup>11</sup> We are just using the fact that  $\phi^{-1}(1)$  is a *filter*, when  $\phi \neq 0, 1$  is multiplicative.



Thus a multiplicative coevent is simply a polynomial that reduces to a monomial (Graphically it is a single simplex), and the support of  $\phi$  is  $F$  itself:

$$F = \text{supp}(\phi) . \tag{2}$$

Finally, remember that we want  $\phi$  to be as nearly homomorphic as it can be. Recalling that  $\phi$  would be a homomorphism if its support were reduced to a single atom, let us postulate that  $F = \text{supp} \phi$  is as small as possible. Our final axioms for this scheme are then:

(iii) no precluded  $A \in \mathfrak{A}$  contains  $F$  as a subset

(iv)  $F$  is minimal subject to (iii)

(Axiom (iii) requires that  $F$  meet the complement of every precluded subset.) I will write  $\widehat{\mathfrak{A}}$  for the set of all coevents that satisfy these axioms. The elements of  $\widehat{\mathfrak{A}}$  are thus the “possible realities”, or “possible bundles of attributes of reality”, depending on how we think about a coevent.

Now consider a two-slit diffraction experiment distilled down to a set of four trajectories joining either of two “apertures”,  $a_1, a_2$  to either of two “detector locations”  $\ell_1, \ell_2$ . (We don’t include any actual detectors.) Calling  $s$  the source, we have

$$\Omega = \{sa_1\ell_1, sa_1\ell_2, sa_2\ell_1, sa_2\ell_2\} \equiv \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} .$$

Suppose that  $\gamma_1$  and  $\gamma_3$  interfere destructively. To each  $\gamma_i$  corresponds an amplitude  $\alpha_i$  (assuming unitary quantum mechanics), and, respecting unitarity, we may take these amplitudes to be

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

The only precluded event is then  $\gamma_1 + \gamma_3$ , which I have written using the algebraic notation introduced earlier. (Regarded as a subset of  $\Omega$  this event would be  $\{\gamma_1, \gamma_3\}$ , which, strictly speaking, should be written as  $\{\gamma_1\} + \{\gamma_3\}$ . But the distinction disappears when one identifies  $\gamma_i$  with  $\{\gamma_i\}$ , as we are doing.) In order to be preclusive,  $\phi$  must therefore satisfy  $\phi(\gamma_1 + \gamma_3) = 0$ .

In the multiplicative scheme,  $F = \phi^{-1}(1)$  must not be contained within  $\gamma_1 + \gamma_3$  (axiom (iii)). Hence it must contain either  $\gamma_2$  or  $\gamma_4$ . For example,  $F = \gamma_2 + \gamma_3$  will do, as will  $F = \gamma_4 + \gamma_1 + \gamma_3$ . Clearly the *minimal* sets of this sort are just  $\gamma_2$  and  $\gamma_4$  themselves, corresponding to the coevents  $\gamma_2^*$  and  $\gamma_4^*$ . These two coevents are the members of  $\widehat{\mathfrak{A}}$ , and both are purely classical. In each, the particle goes through a single slit and continues on to the “bright spot”, avoiding the “dark spot”.

A more interesting example is three-slit diffraction, as described in [1]. For simplicity, let’s limit ourselves to three trajectories,  $a, b, c$ , all meeting at a common “detector location”, and let the corresponding amplitudes be respectively 1, 1,  $-1$ . Now both  $a + c$  and  $b + c$  are precluded, whence no classical coevent can be preclusive, since  $a + c$  and  $b + c$  together cover all of  $\Omega$ .<sup>12</sup> Classical logic leads to an impasse in this case. In the multiplicative scheme,  $F = \phi^{-1}(1) = \text{supp}(\phi)$  must contain both  $a$  and  $b$  in order to avoid falling within either  $a + c$  or  $b + c$ . The only possibilities are  $F = a + b$  and  $F = a + b + c$ , of which only the former is minimal. Hence  $\widehat{\mathfrak{A}}$  contains a single coevent in this example, namely  $\phi = (a + b)^* = a^*b^*$ ; and for it,  $\phi(a) = \phi(b) = \phi(c) = \phi(a + c) = \phi(b + c) = 0$ ,  $\phi(a + b) = \phi(a + b + c) = 1$ . Thus for example, the answer to “Does the particle go through slit  $a$ ?” is “No”, but the answer to “Does it go through *some* slit?” is “Yes”. (One could perhaps imagine these questions in terms of idealized “observations” that discover no more or less than the question asked for; but of course it would be wrong to identify such “observations” with any ordinary physical operations except in very special cases.) In this example, the important thing to notice is that there *is* a viable coevent — only one in this case, but nevertheless enough to avoid the untenable conclusion that classical logic would have reached.

As a final example, consider one that figured heavily in an earlier interpretation of the path-integral [3]. Here we have two variables  $A$  and  $B$ , each of which can take only the

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<sup>12</sup> Proving this by computation provides good practice in the algebraic notation (even though the set-theoretic notation happens to be considerably simpler in this case). We have in general  $X \cup Y = X + Y + XY$ . Hence  $(a + c) \cup (b + c) = (a + c) + (b + c) + (a + c)(b + c) = a + b + 2c + ab + ac + cb + c^2 = a + b + 0 + 0 + 0 + 0 + c = a + b + c = \Omega$ .

value  $+1$  or  $-1$ . If we write  $A$  for the event that  $A = 1$  and  $A'$  for the event that  $A = -1$ , and similarly for  $B$ , then our four-element sample space can be written as

$$\Omega = AB + AB' + A'B + A'B' .$$

Assume that  $A$  and  $B$  are perfectly correlated, in the dynamical sense that  $\mu(AB') = \mu(A'B) = 0$ . From this it follows as well that  $\mu(A + B) = \mu(AB' + A'B) = 0$ , assuming that  $\mu$  is strongly positive.<sup>13</sup> So our precluded events are  $AB'$ ,  $A'B$ , and  $A + B$ . (Notice that  $A + B$  is the event that  $A$  and  $B$  are *anti*-correlated.) With our sample space of 4 elements, we have  $|\Omega| = 4$ ,  $|\mathfrak{A}| = 2^{|\Omega|} = 16$ ,  $|\mathfrak{A}^*| = 2^{|\mathfrak{A}|} = 65536$ , where  $|\cdot|$  denotes cardinality. Of these 65 thousand or so coevents, sixteen are multiplicative, since there are  $2^4 = 16$  subsets of  $\Omega$  to play the role of  $F = \text{supp}(\phi)$ . Which of these sixteen coevents are preclusive, and which of the preclusive ones are minimal? Reasoning as before, we see that  $F$  must contain either  $AB$  or  $A'B'$ . But  $\phi = (AB)^*$  is already preclusive, as is  $\phi = (A'B')^*$ . Hence these are the only minimal preclusive multiplicative coevents. Once again, we've reached the classical solution:

$$\widehat{\mathfrak{A}} = \{(AB)^*, (A'B')^*\} .$$

Indeed this had to happen, because the pattern of preclusions was classical in the sense that it could have arisen from a classical probability-measure  $\mu$ . Equivalently, every subset of a precluded set was also precluded. One can prove that when this happens the minimal  $\phi$  are just the classical coevents  $\gamma^*$ , for non-precluded  $\gamma$ . Hence the multiplicative scheme, like the linear one, reproduces classical logic when quantal interference is absent.

Although this example of “ $A$ - $B$ -correlations” turned out to be fairly trivial in the end, it is important because it illustrates how “anhomomorphic inference” works. Classically we can conclude from “ $A + B$  is false” that exactly one of  $AB$ ,  $A'B'$  is true. Consequently, if  $A$  is true then  $B$  must be true and  $B'$  false (together with  $A'$ ). None of these deductions is guaranteed a priori in anhomomorphic logic. But our analysis of the above example shows that if not only  $A + B$ , but also  $A'B$  and  $AB'$  are false (which classically would have

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<sup>13</sup> Strong positivity of the decoherence functional, as defined in [4], holds automatically in unitary quantum mechanics. Via an analog of the Schwarz inequality, it implies  $\mu(A \cup N) = \mu(A)$  whenever  $N$  has measure zero and is disjoint from  $A$ .

followed from  $A + B$  false), then classical logic does apply, whence the truth of  $A$  does imply that of  $B$  and conversely. [proof: if  $\phi(A) = 1$  then  $\phi$  cannot be  $(A'B')^*$  because  $(A'B')^*(A) = 0$  since  $A'B' \not\subseteq A$ . Hence  $\phi = (AB)^*$  (the only other possibility in  $\widehat{\mathfrak{A}}$ ), whence  $\phi(B) = 1$  and  $\phi(A') = \phi(B') = 0$ .]

Finally, let's turn briefly to the “ideal-based” scheme that generalizes the linear scheme of [1]. For lack of space, I will only sketch this scheme and its application to the above examples. Its advantage is its greater flexibility and generality, but its disadvantage is that it is harder to state. The basic idea is that  $\phi$  should be preclusive (as always), and that the members of  $\widehat{\mathfrak{A}}$  should be the “simplest” that suffice to generate  $\mathfrak{A}_0^*$  as an ideal.<sup>14</sup> Here  $\mathfrak{A}_0^*$  denotes the ideal of all preclusive coevents, and “simple” is yet another word meaning “close to homomorphic”. In the present context it seems best to interpret “simplicity” not in terms of the support of  $\phi$ , but rather in terms of — say — the number of *operations* needed to build up  $\phi$  as a polynomial  $p$  in the classical coevents. Thus defined, simplicity is measured by the sum of the degrees of the monomials whose sum is  $p$ . The smaller is this sum, the “simpler” is  $\phi$ . We then want to take for  $\widehat{\mathfrak{A}}$  the generating set of “maximum simplicity”, or perhaps just the unital members of this set. (In detail, there will be different ways to compare the simplicity of coevents and sets of coevents; in addition it could happen that the simplest generating set was not unique. None of these potential ambiguities surfaces in any of our three examples.)

Applying the ideal-based scheme to our three examples, we find the following (limiting ourselves to the generators that are unital):

*2-slit:*  $\widehat{\mathfrak{A}} =$  same as above.<sup>15</sup>

*3-slit:*  $\widehat{\mathfrak{A}} = \{a^* + b^* + c^*, a^*b^*\}$

*A-B-correlations:*  $\widehat{\mathfrak{A}} =$  same as above

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<sup>14</sup> An ideal in an algebra is a linear subspace of the algebra which is closed under multiplication by arbitrary algebra elements. A subset  $S$  of an ideal  $I$  generates it if  $I$  is the smallest ideal including  $S$ . It is not difficult to verify that the preclusive coevents form an ideal.

<sup>15</sup> There is also a non-classical generator,  $a^* + b^*$ , but it's not unital.

Thus in these examples, the ideal-based scheme differs from the multiplicative scheme only in the three-slit example, where it yields a second coevent, namely the unique coevent produced by the linear scheme of [1]. With respect to this coevent,  $a$ ,  $b$  and  $c$  all become true, while  $a + b$  becomes false.

One pleasant feature of the ideal-based scheme is that (as one can prove) if an event  $A \in \mathfrak{A}$  is not precluded then there exists  $\phi \in \widehat{\mathfrak{A}}$  for which  $A$  happens ( $\phi(A) = 1$ ).

REMARK Notice that anhomomorphic logic shifts the emphasis from individual histories  $\gamma \in \Omega$  to the algebra of predicates  $\mathfrak{A}$ . This shift could seem like a retreat from reality, but it is perhaps natural from a “dialectical” starting point that takes the whole as prior to its parts. To “unexamined materialism”, reality is simply a single history  $\gamma \in \Omega$ , what I called earlier a “formal trajectory”. Yet such an element only begins to take shape when we subdivide the whole into its (spatio-temporally separated) parts. To the extent that the subdivision remains to be completed, as it always must, we recognize only a “coarse-grained world” corresponding to the subalgebra of  $\mathfrak{A}$  generated by the coarse-grained variables. Perhaps such musings lend a greater “dignity” to the elements of  $\mathfrak{A}$ , as being in some sense logically prior to those of  $\Omega$ . Anhomomorphic logic is also “dialectical” in a second, more obvious sense: it “admits contradictions”.

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