Toward a "fundamental theorem of quantal measure theory"*

Rafael D. Sorkin

Perimeter Institute, 31 Caroline Street North, Waterloo ON, N2L 2Y5 Canada
and
Department of Physics, Syracuse University, Syracuse, NY 13244-1130, U.S.A.

address for email: rsorkin@perimeterinstitute.ca

Abstract

We address the extension problem for quantal measures of path-integral type, concentrating on two cases: sequential growth of causal sets, and a particle moving on the finite lattice $\mathbb{Z}_n$. In both cases the dynamics can be coded into a vector-valued measure μ on Ω, the space of all histories. Initially μ is defined only on special subsets of Ω called cylinder-events, and one would like to extend it to a larger family of subsets (events) in analogy to the way this is done in the classical theory of stochastic processes. Since quantally μ is generally not of bounded variation, a new method is required. We propose a method that defines the measure of an event by means of a sequence of simpler events which in a suitable sense converges to the event whose measure one is seeking to define. To this end, we introduce canonical sequences approximating certain events, and we propose a measure-based criterion for the convergence of such sequences. Applying the method, we encounter a simple event whose measure is zero classically but non-zero quantally.

Keywords and phrases: quantal measure theory, path integral, measure theory, descriptive set theory.

1. Introduction

In order to define area, even for something as simple as a disk of unit radius, one needs to invoke an extension theorem. In a systematic development [1] of plane-measure, one

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begins by defining the measure μ of an arbitrary rectangle, and one then seeks to extend the set-function μ unambiguously to subsets of the plane that can be made from rectangles via countable processes of union and complementation (these sets comprising the σ-algebra generated by the rectangles). The unit disk is such a subset, and (if we take it to be open) it obviously can be built up as the disjoint union of a countable family of rectangles. But this can be done in an infinite number of different ways, and one needs to know that the net area of the rectangles is always the same, no matter which decomposition one chooses and no matter in which order one chooses to perform the resulting sum. The theorem that guarantees this consistency is known as the Kolmogorov-Carathéodory extension theorem, but it might also be called the "fundamental theorem of classical measure theory". Not only is it used to construct Lebesgue measure, but it plays a central role in defining stochastic processes like the Wiener process, a mathematical model of Brownian motion that also describes the Wick rotated path-integral for a non-relativistic free particle on the line.

In this sort of application, one is dealing with a probability-measure on a space of paths or more generally "histories", and the possible values of μ are therefore positive real numbers between 0 and 1. However, when one seeks to define a genuine path integral in real time (as opposed to Wick-rotated, imaginary time), one encounters complex amplitudes that can be arbitrarily large and of any phase. Once again, there are specially simple sets of paths, analogs of the rectangles, called "cylinder sets", from which the more general sets of interest can be built up, but the sums that arise in this case no longer converge absolutely. In technical terms the complex measure one is trying to extend is not of bounded variation, and the available extension theorems cannot be used [2].

The problems that one faces vary, depending on context. There are "ultraviolet" problems springing from the infinite divisibility of the paths or "histories" one is trying to sum over, and there are "infrared" problems that arise in connection with histories that are unbounded in time. By limiting ourselves to spatio-temporally discrete processes we nullify the former problems, and that will be the context of the rest of this paper, where we will encounter only discrete histories like those that occur in a random walk. It will thus be only issues of infinite time that will occupy us.

The concrete instances we will consider will be of one of two types, which we can characterize by the kind of "sample space" or "history space", Ω, on which one builds. The first instance arises in the context of quantum gravity and more specifically within the causal set programme. There the discreteness reflects the finiteness of Planck's constant, and the underlying physical process is a kind of "birth" or "accretion" process by means of which the causal set is built up or "grows". The corresponding sample-space of "completed" causal sets consists of all the countable, past-finite partial orders P; and one is seeking to define a certain type of vector-valued measure μ on it. (The dynamics determines μ only up to a unitary transformation. The object of direct physical interest is not μ itself but a certain scalar-valued set-function belonging to the class of strongly positive decoherence functionals or quantal measures on Ω. However, any such a functional can be represented [3] as a measure on Ω which is valued in some Hilbert space $\mathfrak{H}$.) In the second type of example, the elements of Ω will be discrete-time trajectories moving in a lattice that will be either the integers modulo n ($\mathbb{Z}_n$) or just the integers as such ($\mathbb{Z}$). These examples correspond to a widely studied class of processes known as "quantal random walks", but for us they will be important primarily as simplified analogs of causal set growth processes. In that role, they are particularly illuminating because their sample spaces are essentially the same ones that "descriptive set theory" investigates.

How certain are we, though, that the quantal measures in these all instances really need to be defined in a new way? With the lattices, the dynamical laws in question are those of the evolution generated by a unitary operator or "transfer matrix". In their path-integral formulation, such unitary laws inevitably lead to measures of unbounded variation [3], and the theorems of the Kolmogorov-Carathéodory type are thus guaranteed to fail. In the more important, causal set case however, there remains some doubt, especially given the anticipated breakdown of unitary evolution in that case. The only fully developed dynamics one has for causal sets is that of the classical sequential growth (CSG) models, which in themselves are not quantal in nature. For them, the usual extension theorems do suffice because one is dealing with a classical probability measure [4]. But if one complexifies the parameters of a CSG model, one obtains straightforwardly a family of quantal measures (decoherence functionals) which are in general neither unitary nor of bounded variation [3]. Although none of these complexified CSG dynamics is likely to exhibit quite the type of interference required by quantum gravity, the fact that the measures that arise are not of bounded variation suggests that this might turn out to be a general feature of quantal causal sets, just as it is a general feature of quantal path integrals in other contexts. Nevertheless it's worth keeping in mind the possibility that the physically appropriate quantal measures for causal set dynamics will turn out to be σ-additive in the traditional sense. Were that to happen, quantum gravity would have revealed itself to be more tractable mathematically than the nominally much "simpler" non-relativistic free particle! The problems addressed in the present paper would then be pseudo-problems, as far as quantum gravity went.

In our current state of ignorance, however, it seems prudent not to count on so much good fortune. And besides, one might still like to have a well defined path-integral for systems like the free particle, without having to embed them in a full-blown theory of quantum gravity. What, then, can one do when bounded variation fails? As urged in reference [3], such a failure need not be the end of the story, because in a concrete physical situation, the space of histories has more structure than what is available in an arbitrary measure space. Indeed, physicists routinely work with infinite sums and integrals that converge only conditionally. Typically one introduces a "cutoff" or integrating factor in a manner mandated by physical considerations, in effect doing the sums or integrals in a particular order so that their convergence need not be absolute. In the case of the planar disk, for example, instead of expressing it as a disorganized sum of an infinite number of rectangles, one might think to employ a definite sequence of approximations, each consisting only of rectangles bigger than a certain size ϵ. The area would then be given by the $\epsilon \to 0$ limit of these approximations.

In our situation, one can attempt something similar by considering "late-time" cylinder sets to be "finer" than "early-time" ones. In order to implement this idea we will seek first of all, for any given set $A \subseteq \Omega$ of histories or paths, a "canonical" sequence of approximations A_n to A in terms of cylinder sets, and this sequence should be as near to unique as feasible. Then, given such a sequence, we will try to decide what further convergence properties it ought to have in order that we can form a limit $\mu(A)$ of the individual $\mu(A_n)$ and consistently attribute this limit to A as its quantal measure.

In what follows, we take only a few steps in the direction indicated, pointing out along the way various pitfalls that one needs to avoid. Hopefully this can at least illustrate the kind of approach one might take to the extension problem for quantal measures. Sections 6 and 7 are the heart of the paper. In section 6, we will define canonical approximations for a limited class of *events*, as (sufficiently regular) subsets $A \subseteq \Omega$ are normally designated. We will then introduce, in section 7, a convergence criterion for an approximating sequence A_n, and we will prove that the resulting extension of μ is additive for disjoint unions of open sets. Limited as our approximation scheme will be, it will at least embrace the type of event A which is most important for the sake of causal sets, namely the *covariant stem-event*. Among all the sets of histories to which one might wish to assign a measure, the only indispensable ones are these. Without their aid it would be nearly impossible to produce a generally covariant dynamical scheme in any useful sense [5] [4].

An appendix lists some of the symbols used in the body of the paper.

Contents

2. Sample-spaces and amplitudes for causal sets and the 2-site hopper

causal sets

A causal set (or *causet*) [6] [7] [8] [9] [10] in its most general conception can be any locally finite partial order or *poset*, but in the context of the dynamics of sequential growth and quantal cosmology no element of the causet will possess more than a finite number of ancestors. For present purposes we may thus define a causet as a *past-finite countable poset*, i.e. a countable (possibly finite) set of *elements* endowed with a transitive, acyclic order-relation, $\prec$, which I will also take to be irreflexive. These concepts are exposed in greater detail in [11], where the notion of sequential growth is also explained. Here I will just summarize the main definitions and introduce the notation we will use.

A sequential growth process proceeds as a succession of "births" of new elements, and in this sense is never ending. If however, one idealizes it has having "run to completion", it will have produced a *completed causet* as defined above: a countable set of elements, each having a finite number of predecessors or *ancestors* but a possibly infinite number of descendants. The set of all such causets constitutes the natural sample-space Ω for this process. Actually, one must distinguish here two distinct sample spaces, which one may call Ω^{gauge} and $\Omega^{physical}$. The latter, which in some sense is the true sample-space, consists of *unlabeled* causets, or equivalently isomorphism equivalence classes of causets. The former, which I'll normally denote simply as Ω, then consists of the *naturally labeled* causets, a natural labeling being a numbering, $0, 1, 2, \ldots$ of the elements which is compatible with the defining order $\prec$: if $x \prec y$ then y carries a bigger label than x. Here again, one of course really intends isomorphism equivalence classes of labeled causets (or if you like the elements could be taken to be the integers themselves in this case).

The labels record the order of the respective births, and what is most important for us here is that this order is supposed to be fictitious in the same sense as is a choice of coordinate system for a continuous spacetime is fictitious. The physically meaningful or *covariant* events will thus correspond to subsets of $\Omega^{physical}$, whereas the measure μ defining the growth process is in the first instance defined on Ω^{gauge}. But even a very simple subset of $\Omega^{physical}$, even a singleton, will equate to a much less accessible subset of Ω^{gauge}, namely the subset obtained by taking every possible natural labeling of every member of the original subset. Thus arises the need for an extension of μ that will assign well defined measures to such "covariant" subsets of Ω^{gauge}. Unlike for the example of the hopper to be discussed next, this is not just a matter of convenience if one wants to be in a position to ask truly label-independent questions about the causet.

Henceforth in this paper all causets will be labeled unless otherwise specified. (In reference [4] the true or "covariant" sample space was denoted simply by Ω, while its labeled counterpart was $\widetilde{\Omega}$. Here however, it seems simpler to use Ω for the latter, since it is the space we will usually be dealing with.)

In reference [4], the measures defining the CSG dynamical models were defined rigorously by extending a probability measure given originally on the space $\mathfrak{Z}$ of *cylinder events* (or cylinder sets), where a cylinder event $\mathrm{cyl}(c) \in \mathfrak{Z}$ is by definition the set of all completed causets containing a given, naturally labeled, finite causet c. A finite causet will also be called a *stem* and on occasion a "truncated history". In conjunction with these definitions, let us also define $\Omega(n)$, the space of all naturally labeled causets of n elements, and $\mathfrak{Z}(n)$ or $\mathfrak{Z}_n$, the space of cylinder events of the form $\mathrm{cyl}(c)$ for $c \in \Omega(n)$. The cylinder sets comprise what is called a "semiring" of sets in the sense that given any two cylinder sets, Z_1 and Z_2, their intersection, $Z_1 Z_2 \equiv Z_1 \cap Z_2$, is also a cylinder event, and their difference $Z_1 \backslash Z_2$ is the disjoint union of a finite number of cylinder events. In fact the cylinder events form an especially simple kind of semiring, because any two of them are either disjoint or nested.

To rehearse the definition of the CSG models in general would take us too far afield, but the special case of "complex percolation" is simple enough to be given here in illustration of the general scheme. The vector measure μ is determined in this case by a single complex parameter p, and it takes its values in a one-dimensional Hilbert space that we may identify with $\mathbb{C}$, so that $\mu(A)$ is itself just a complex number. Now let $c \in \Omega(n)$ be a labeled causet of n elements and let $Z = \mathrm{cyl}(c)$ be the corresponding cylinder set. Then $\mu(Z) = p^L (1-p)^I$, where $L = L(c)$ is the number of links in c and $I = I(c)$ is the number of incomparabilities. Here an incomparability is simply a pair of unrelated elements, and a link is a causal relation, $x \prec y$, which is "nearest neighbor" in the sense that there exists no intervening z for which $x \prec z \prec y$.

Observe now that the collection of naturally labeled finite causets, i.e. the space $\bigcup_n \Omega(n)$, has itself the structure of a poset in a natural way. Indeed this poset is actually a tree $\mathfrak{T}$, because its elements are labeled. (The corresponding structure formed by the unlabeled stems is a more interesting poset called *poscau* in reference [11].) Clearly, a particular realization of the growth process, or equivalently the resulting completed causet in Ω, can be conceived of as an upward path through this tree. An analogous conception will be possible for the two site hopper, and in this guise seems to be exploited heavily in descriptive set theory [12][13]. (See figures 1 and 2.)

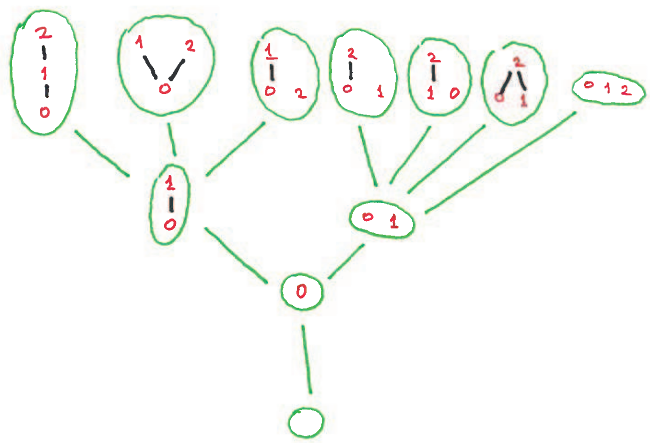

Figure 1. The first 4 levels of the tree $\mathfrak{T}$ of naturally labeled causets

Finally, let us define an *event algebra* to be a family of subsets of the sample space Ω closed under the operations of intersection and complementation. An event algebra is thus a Boolean algebra or "ring of sets". To the extent it can be achieved, one normally wants the domain of μ to be such an algebra, because for example, if the events "A happens" and "B happens" are of interest, then so also is the event "either A or B happens". The cylinder sets $\mathfrak{Z}$ do not themselves form an algebra, but the family $\mathfrak{S}$ of finite unions of cylinder sets does. It is in fact $\mathfrak{R}(\mathfrak{Z})$ the Boolean algebra generated by $\mathfrak{Z}$. In all cases of interest μ will automatically extend uniquely from $\mathfrak{Z}$ to $\mathfrak{S}$, yielding a finitely-additive measure thereon. The space $\mathfrak{S}$ thus constitutes a minimum domain of definition for the vector-measure μ. The question then will be how far μ can be extended beyond $\mathfrak{S}$ into the σ-algebra generated thereby, the hope being that the enlarged domain $\mathfrak{A}$ will itself be an event algebra, and that it will contain enough events so that, at a minimum, the physically most important questions will become well posed. (Some noteworthy instances of covariant questions/events will be discussed in the next section.)

n-site hopper

By "2-site hopper" I mean the formalization of a particle residing on a 2-site lattice and at each of a discrete succession of moments either staying where it is or jumping to the other site [14]. For definiteness, I will assume that the moments are labeled by the natural numbers, the sites by $\mathbb{Z}_2$, and that at moment 0 the hopper begins at site 0. The definitions of sample space, cylinder event, etc. are closely analogous to those given above for causets, and references to them should be understandable without their formal definitions, which I will postpone until after the transition amplitudes have been specified. The full course of the motion, idealized as having run to completion, will be called a *path* or "history". Notice that, modulo the small ambiguity in how a real number can be expressed as a "binary decimal", each such path can be identified uniquely with a point in the unit interval $[0, 1] \subseteq \mathbb{R}$.

Aside from a simpler sample space than in the causet case, the hopper offers us in addition a fuller illustration of the problems of defining the vector-measure corresponding to a path integral. Unlike the former case, where the correct choice of quantal amplitudes is only conjectural, there exists for the hopper a choice that can be interpreted as a straightforward discretization of the Schrödinger dynamics of a non-relativistic free particle moving on a circle (cf. [15]).

These amplitudes can be understood more easily if one sets them up, not just for two sites, but for the more general case of the circular lattice $\mathbb{Z}_n$ ("n-site hopper"). Perhaps they will look most familiar if presented as the unitary evolution operator or "transfer matrix" analogous to the propagator that solves the Schrödinger equation in the continuous case. To that end, let $x \in \mathbb{Z}_n$ be the location of the particle at some moment t, let x' be its location at the next moment $t' = t + 1$, and write for brevity $\exp(2\pi i z) \equiv \mathbf{1}^z$. The amplitude to go from x to x' in a single step is then

$$\frac{1}{\sqrt{n}}\,\mathbf{1}^{(x-x')^2/n}$$

for n odd and

$$\frac{1}{\sqrt{n}}\,\mathbf{1}^{(x-x')^2/2n}$$

for n even. For example, for $n = 6$ and with $q = \mathbf{1}^{1/12}$, the (un-normalized) amplitudes to hop by 0, 1, 2 or 3 sites are respectively $q^0 = 1$, $q^1 = q$, q^4, and $q^9 = -i$. For the 2- and 3-site hoppers, the above amplitudes are particularly simple, yielding for $n = 3$ the transfer matrix

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \omega & \omega \\ \omega & 1 & \omega \\ \omega & \omega & 1 \end{pmatrix} \qquad\qquad (\omega = 1^{1/3})$$

and for $n = 2$ the transfer matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\ . \tag{1}$$

From these expressions and the definition of the decoherence functional it is not hard to carry out the construction of the equivalent vector measure along the lines of [3]. In the simplest case of two sites, which will be our main example herein, μ is valued in a two-dimensional Hilbert space $\mathbb{C}^2$ and, with a convenient choice of basis vectors, can be expressed as follows. Let $(0\,x_1\,x_2\,x_3\ldots x_m)$ be a truncated path and let $Z \subseteq \Omega$ be the corresponding cylinder event. Then $\mu(Z) \equiv |Z\rangle$ will be the two-component complex vector v_α where (no summation implied) *

$$v_\alpha = (U^{-m})_{\alpha x_m} U_{x_m x_{m-1}} \cdots U_{x_3 x_2} U_{x_2 x_1} U_{x_1 0} \ , \tag{2}$$

U being the unitary matrix of equation (1). Notice incidentally that U^j is periodic with period 8 and is very easy to compute explicitly, since $U^4 = -1$ while $U^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ is also very simple.

Finally the formal definitions for the 2-site hopper. A truncated history, the counterpart of a finite causal set, is for the hopper an initial segment of a path, for example $(0, 1, 1, 0, 1)$. (Recall our boundary condition that all paths begin at zero.) The set of all such truncated histories which have length n when the initial 0 is omitted will be $\Omega(n)$, and the corresponding cylinder events will be the elements of $\mathfrak{Z}_n$. The semiring $\mathfrak{Z}$ will be the union of the $\mathfrak{Z}_n$. For example, $\text{cyl}(0, 1, 1, 0, 1) \in \mathfrak{Z}_4$ is the set of all completed paths of the form $(0, 1, 1, 0, 1, x_5, x_6, \cdots)$. Exactly as above, $\mathfrak{S}$ will be the Boolean algebra generated by $\mathfrak{Z}$. One can check straightforwardly that μ, as defined by (2), extends uniquely and consistently to each $\mathfrak{S}_n$ and therefore to $\mathfrak{S}$ as a whole. Again, the truncated histories can be construed as the nodes of a tree $\mathfrak{T}$, the "branches" or "edges" being given by extension of path. (See Figure 2.) For example, there will be an edge from $(0, 1, 1, 0)$ to $(0, 1, 1, 0, 1)$.

* Another notation for v_α could be $\langle \alpha | 0\, x_1\, x_2\, x_3 \ldots x_m \rangle$

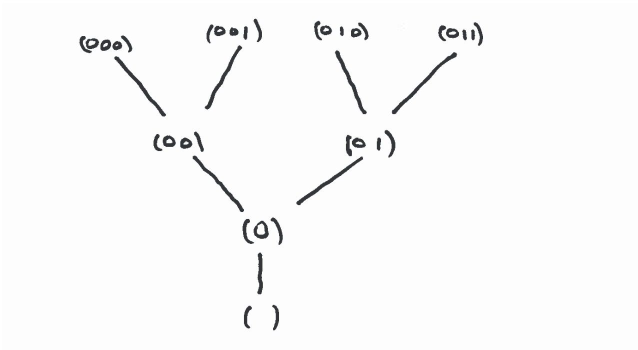

Figure 2. The first 4 levels of the tree $\mathfrak{T}$ for the 2-site hopper

In what follows, it will sometimes be enlightening to consider hopper-paths on the infinite lattice $\mathbb{Z}$. In that case the paths will be restricted to move no more than one site per step ("random walk"), in order that the resulting tree $\mathfrak{T}$ continue to have a finite number of branches emanating from each node.

For an extensive discussion of the quantal 2-site hopper see [14]. For more general sorts of quantal random walks see [16].

3. Some events whose measures one would like to define

The event algebra $\mathfrak{S}$ generated by the cylinder events supplies enough events to allow one to ask any question[†] about the process under consideration, as long as it doesn't refer to happenings arbitrarily far into the future. But often one does not want to be bound by this limitation, especially since in the causet case, the "time" referred to contains a large element of gauge, as explained above.

To illustrate how "infinite-time" events enter the story, let us dwell on a few examples, beginning with the n-site hopper. Perhaps the simplest and most familiar example of this kind is the event R of *return*, which occurs if and when the particle returns to its starting point at some later time. This event, in other words, is the set of all paths $(0\,x_1\,x_2\,\ldots)$ for which one of the $x_i = 0$. Plainly R is not in $\mathfrak{S}$, because the return, although it must occur at a finite time if it occurs at all, can take place arbitrarily late. For a *classical* hopper on

† The words "event" and "question" are in a certain sense synonyms. To an event $A \subseteq \Omega$ corresponds the question "Does A happen?". Note in this connection that (except in the classical case) it would lead to confusion if one read "A happens" as "the path is an element of A", cf. [17].

a finite lattice, one knows that $\mu(R)$, the measure of the return-event (which classically is its probability), is unity, but to express this fact directly, we need R to be in the domain of μ. Of course, we could circumvent any direct reference to R by introducing the finite-time event R_n that the particle returns on or before the n^{th} step. Instead of asserting that $\mu(R) = 1$, we could then say "The sequence $\mu(R_n)$ converges to 1 as $n \to \infty$". Plainly, the first formulation is simpler and less cumbersome to work with. Notice in this case that not only at the level of the measures, but even at the level of the events themselves, R is the limit of the R_n in a natural sense, since the latter are *nested* and "increase monotonically to R". That is, one has $R_1 \subseteq R_2 \subseteq R_3 \cdots$, with R itself being the union of the R_n, or logically speaking their "disjunction". Were μ a classical measure, this would guarantee convergence of the $\mu(R_n)$ and consistent extension of the domain of μ to include the event R; in the quantal case it guarantees nothing.

A similar event to "return", but one which is related even less directly to any cylinder event, is the event R^∞ that the particle visits $x = 0$ infinitely often. This event also has a well-defined probability of unity in the classical case. Since, however, it cannot come to fruition at any finite time, it cannot — unlike the event R of simple return — be expressed as a union of cylinder sets or other members of $\mathfrak{S}$. Instead it is a countable intersection of events, each of which is a countable union of events in $\mathfrak{S}$. For example, let $E(j, k)$ for $j < k$ be the event that $x_k = 0$. Then $R^\infty = \bigcap_j \bigcup_k E(j, k)$. (In words: for each moment j there is a later moment k at which the particle visits the origin.[♭]) To give meaning to $\mu(R^\infty)$ by prolonging the initially defined measure with domain $\mathfrak{S}$ one would thus have to think in terms of a limit of limits.

As a third example (restricted this time to one of the lattices, $\mathbb{Z}$ or $\mathbb{Z}_n$ with $n > 4$), consider the event that the particle visits $x = 3$ but never reaches $x = 5$. Intermediate between the two previous examples in its remoteness from $\mathfrak{S}$, this event is naturally expressed as the set-theoretic difference of two limits of finite-time events, the first being, naturally the event F that the particle reaches $x = 3$ and the second G that it reaches $x = 5$. Just as with the return event R, the event $F \backslash G$ is, in a well defined sense to which

[♭] One can often arrive at such combinations by beginning with a formal statement of what it means for the event to happen. In this case one might first write down what it means for R^∞ *not* to occur: $(\exists n_0)(\forall n > n_0)(x_n \neq 0)$, and then negate it to obtain $(\forall n_0)(\exists n > n_0)(x_n = 0)$. The nested combination of unions and intersections is basically just a translation of this second statement into set-theoretic language.

we will return below, a limit of events in $\mathfrak{S}$, but it is not simply the union or intersection of a monotonically increasing or decreasing sequence.

It is useful at this point to introduce some further notation to help in discussing the types of events we have just met. Let $\mathfrak{X}$ be any collection of subsets of Ω closed under pairwise union and intersection. Then $\bigvee \mathfrak{X}$ will be the family of events of the form $\bigcup_{n=1}^{\infty} X_n$, where $X_n \in \mathfrak{X}$ and $X_1 \subseteq X_2 \subseteq X_3 \cdots$. It is easy to see that $\bigvee \mathfrak{X}$ is also closed under union and intersection, also that it would not change if we dropped the monotonicity condition, $X_1 \subseteq X_2 \subseteq X_3 \cdots$. In words, the members of $\bigvee \mathfrak{X}$ are the unions of monotonically increasing events in $\mathfrak{X}$. For the intersections of monotonically decreasing events in $\mathfrak{X}$, I will write dually $\bigwedge \mathfrak{X}$. And for the Boolean algebra generated by $\mathfrak{X}$, I will write, as above, $\mathfrak{R}\,\mathfrak{X}$ or $\mathfrak{R}(\mathfrak{X})$. Our first example, "return", is then an element of $\bigvee \mathfrak{S}$, our second of $\bigwedge \bigvee \mathfrak{S}$, and our third of $\mathfrak{R} \bigvee \mathfrak{S}$, while for $\mathfrak{S}$ itself, we have $\mathfrak{S} = \mathfrak{R}(\mathfrak{Z})$.

Turning now to events for causal sets, we will encounter some types very similar to those just discussed. Foremost in importance are the unlabeled stem-events mentioned earlier. Given two causets c and c', of which the first is finite, we say that c' admits c as a *stem* (or "partial* stem") if c' contains a downward-closed subset that is isomorphic to c. In the context of sequential growth, this can also be expressed by saying that it might have happened that elements of c were all born before any of the remaining elements of c'. A stem thus generalizes the notion of "initial segment". The *stem-event* 'stem(c)' is then the set of all $c' \in \Omega$ which admit c as a stem. The stem c that enters this definition is taken to be unlabeled, because our purpose is to produce a label-independent or "covariant" event. It is evident that stem(c) is indeed covariant in this sense, since the condition that defines it does not refer to the labeling of c'.

The importance of the stem events physically is that essentially any covariant question that we care to ask about the causet can in principle be phrased in terms of stem-events. The precise result proven in [4] is that every covariant event is equal, up to a set of measure zero, to a member of the σ-algebra generated by the stem-events. One can also prove that any covariant event which is open in the topology of section 4 below is a countable union of stem events, a purely topological result that holds independently of any assumption about the measure μ. Ideally then, the domain of μ would embrace the whole σ-algebra

* One can also define "full stems" [11], but there is no special reason to consider them here.

generated by the stem-events. At a minimum, one would hope that it would embrace the stem-events themselves.

Now the event 'stem(c)' does not belong to the domain $\mathfrak{S}$ on which μ is initially defined, because it is not a finite-time event when referred to "label-time". If it were, then there would exist some integer N such that if the growing causet c' admitted c as a stem, it would already admit it as soon as the first N elements had been born. But in fact there is nothing in principle to stop the stem in question appearing at an arbitrarily late stage of the growth process. Evidently, the situation is like that of the hopper event "return". Based on this analogy, one would expect the stem events to be found in $\bigvee \mathfrak{S}$, and so they are, as follows directly from the fact that any stem-event is a union of cylinder sets:

$$\mathrm{stem}(c) = \bigcup \left\{ \mathrm{cyl}(\widetilde{b}) \in \mathfrak{Z} \,:\, \widetilde{b} \text{ admits } c \text{ as a stem} \right\} . \tag{3}$$

The problem of extending the vector-measure μ from $\mathfrak{S}$ to $\bigvee \mathfrak{S}$ is thus the most basic one for causal sets.

Starting from the stem-events, one can build up other covariant events, whose occurrence or non-occurrence is of interest for cosmology. The simplest of these is the event that the causet is "originary", meaning that all its elements descend from a unique minimal element or "origin". To say that a completed causet is originary is simply to say that it contains no second minimal element, for which it is necessary and sufficient that it fail to admit the 2-element antichain as a stem. (An antichain is a set of elements which are mutually unrelated or "spacelike" to one another.) Thus the event 'originary' is the complement of the event 'stem(a)', where a is the antichain of two elements. As such it belongs to $\bigwedge \mathfrak{S}$, since as we have seen, the stem events all belong to $\bigvee \mathfrak{S}$, and union turns into intersection under complementation.

If an originary causet represents a certain kind of "big bang" then a causal set containing what is called in the combinatorics literature a *post* describes a "cosmic bounce". (A post is an element of a poset which is spacelike to no element.) In its degree of remoteness from the elementary cylinder events, the post event is comparable to the event of "infinite return" in the case of a random walk, the similarity being even closer if we compare the post event to the *complement* of the infinite return event. In fact both events belong to $\bigvee \bigwedge \mathfrak{S}$, although this is less easy to demonstrate for the post-event than it is in the case of return. To see why it is nevertheless true, imagine watching a succession of births of causet elements, $x_0, x_1, x_2 \dots$ and waiting for a post to be born. If the birth in question is that of element x_n then x_n must have every previous element as an ancestor: $x_j \prec x_n$ for all $j < n$. This renders x_n momentarily a "candidate for becoming a post", but it does not guarantee that x_n will remain a viable candidate forever. In order for that to occur, every subsequent element, $x_{n+1}, x_{n+2}, \cdots$, must arise as a descendant of x_n, i.e. $x_j \succ x_n$ for all $j > n$. By thinking of the post event P in this manner, namely as the set of all sequences of births satisfying the condition that a candidate post appear at some stage n and then not lose its viability at any later stage $m > n$, one can deduce that P belongs to $\bigvee\bigwedge\mathfrak{S}$. The most "covariant" (albeit not the most direct) construction of P as an elt of $\bigvee\bigwedge\mathfrak{S}$ proceeds by first expressing it in terms of stem-events, as we now explain. Proceeding in this manner will also illustrate the thesis that all covariant questions of interest can be expressed in terms of stem-events.

Note first that if x is a post then its exclusive past $T = \{y : y \prec x\}$ is not only a stem, but what has been called a "turtle" [18], meaning in the present context a stem that wholly precedes its complement: $(\forall x \in T)(\forall y \notin T)(x \prec y)$. Some thought reveals that a causet contains a turtle of n elements iff every stem of cardinality $n+1$ has a unique maximal element. Introducing the term *principal* for such a stem, together with the terms n-stem [resp. n-turtle] for a stem [turtle] of n elements, we can say succinctly that *a causet contains an n-turtle iff every $(n+1)$-stem is principal.* Furthermore it's easy to demonstrate that x is a post iff both its exclusive *and* inclusive pasts are turtles (the exclusive past being $\{y \neq x : y \prec x\}$ and the inclusive past being $\{y : y \preceq x\}$). Therefore, in a labeled causet, element x_n is a post iff every stem of either $n+1$ or $n+2$ elements is principal. Let P_n be the event that this happens, then the post event itself is $P = \cup_{n=1}^{\infty} P_n$.

Now let us examine the event P_n more closely. It *fails* to happen iff some $(n+1)$-stem or $(n+2)$-stem fails to be principal. Let $S_1^n, S_2^n, \ldots, S_{K_n}^n$ be an enumeration of all such stems (there being only a finite number of n-stems, for any n), and let $Q_j^n = \mathrm{stem}(S_j^n)$ be the corresponding stem-events. We then obtain P_n in the "manifestly covariant" form $P_n = \Omega \backslash (\cup_j Q_j^n) = \Omega \backslash (\cup_j \ \mathrm{stem}(S_j^n))$. P is thus a countable union of finite Boolean combinations of stem events:

$$P = \bigcup_{n=0}^{\infty} (\Omega \setminus (\bigcup_{j=1}^{K_n} \mathrm{stem}(S_j^n))) \ . \tag{4}$$

If one knew how to take stem events as primitive, P would thus be a rather simple type of event, inasmuch as the inner union only ranges over a finite number of events. But given that the extant dynamical schemes all begin with labeled causets, we will still need to trace everything back to the cylinder events $\mathfrak{Z}$.

First, though, a simple example might be in order, say for $n = 1$. (The event P_0 is just the originary event, which one might not even want to count as a post.) The event P_1 requires that all 2- and 3-stems be principal. The only 2-stem that can occur is thus the 2-chain $(a \prec b)$, while the admissible 3-stems are the 3-chain $(a \prec b \prec c)$ and the "Λ-order" $(a \prec c, b \prec c)$. The stems that must be excluded — those denoted above by S_j^n — are correspondingly the 2- and 3-stems which are not principal: the 2-antichain, the 3-antichain, the "L-order" $(a \prec b, c)$, and the "V-order" $(a \prec b, a \prec c)$. (See figure 3.)

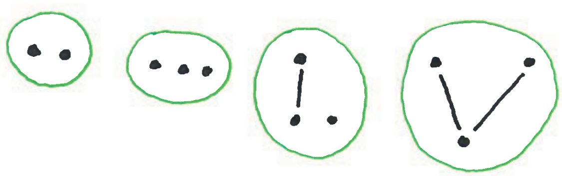

Figure 3. The non-principal 2- and 3-stems.

To complete the demonstration that $P \in \bigvee \bigwedge \mathfrak{S}$, let us return exclusively to labeled causets, observing first that in view of equation (4), it suffices to show that the complement of a finite union of stem-events belongs to $\bigwedge \mathfrak{S}$. (Strictly speaking, given how we have defined the operation $\bigvee$, we also need to convert the outer union in (4) into an *increasing* countable union of events in $\bigwedge \mathfrak{S}$. That this is possible follows readily from the relation (6) of Section 5, which informs us that, when A_n and B_n are both decreasing sequences of sets, then the union of their limits coincides with the limit of the decreasing sequence $A_n \cup B_n$, in consequence of which $\bigwedge \mathfrak{S}$ is closed under finite union, and we can replace a countable union $\cup_n F_n$ of events $F_n \in \bigwedge \mathfrak{S}$ with the increasing union $\cup_n F'_n$, where $F'_n = \cup_{m \le n} F_n$.) To that end, recall that any stem-event A is an increasing union of events in $\mathfrak{S}$. Its complement, $\Omega \backslash A$, is therefore a decreasing intersection of complements of events in $\mathfrak{S}$, each of which is itself in $\mathfrak{S}$ since the latter, being a Boolean algebra, is closed under complementation. Hence, the complement of a stem-event belongs to $\bigwedge \mathfrak{S}$, and the same holds for the complement of a finite union of stem-events, such as occurs in (4).

For reasons that will become clear shortly, it is natural to designate the elements of $\mathfrak{S}$ as *clopen*, meaning "both closed and open" in the sense of point-set topology. [†] The events in $\bigvee \mathfrak{S}$ will then be *open*, those of $\bigwedge \mathfrak{S}$ will be *closed*, and we will have expressed our post-event $P \subseteq \Omega$ as an increasing limit of closed subsets of Ω. Continuing in this vein, more elaborate combinations of the clopen events can be formed, including for example the event that infinitely many posts occur. But the physical relevance of such combinations seems to shrink rapidly as their complexity grows. Indeed, one might feel that, questions of convenience aside, no event more complicated than a finite Boolean combination of stem-events can claim to be indispensable. One might even go farther and call into doubt the status of complementation (negation), leaving unquestioned only those events formed as finite unions and intersections of stem-events.

4. Ω as a compact metric space

open and closed sets

Whenever the idea of convergence plays a role, one can expect, almost by definition, that topology will make an appearance. In the present situation, we are talking about convergence to a given event $A \subseteq \Omega$ of a sequence of approximating events A_n, in the first instance events formed as finite unions of cylinder events and thus belonging to the event-algebra $\mathfrak{S}$. In setting up such a sequence of approximations, we would like, as explained earlier, to regard those cylinder events that specify a greater portion of the history as more "fine grained" than those that specify a lesser portion. This leads very naturally to a definition of distance between histories that makes Ω into a compact metric space [4] [12].

As implemented for causal set growth processes, the definition runs as follows. For each pair of completed labeled causets a, $b \in \Omega$, we set:

$$d(a, b) = 1/2^n \ , \tag{5}$$

where n is the largest integer for which elements $a_0\, a_1\, \cdots a_n$ produce the same poset (with the same labeling) as elements $b_0\, b_1\, \cdots b_n$. It is easy to verify that this yields a metric on Ω; indeed d satisfies a condition stronger than the triangle inequality: for any three causets a, b and c, we have $d(a, c) = \max(d(a, b), d(b, c))$. This "ultrametric" property derives from

[†] In the context of abstract measure theory, the term "elementary sets" was used in reference [1] to refer to events analogous to those of $\mathfrak{S}$.

the tree structure of the space $\mathfrak{Z}$ of cylinder sets (or equivalently truncated histories), as described earlier. The maximum distance between two causets is $1/2$ and occurs when their initial two elements already form distinct partial orders. Notice also that the open balls in this metric are exactly the cylinder sets, with the radius of the ball serving as a measure of "fineness".

One can see without too much difficulty that with this metric, Ω becomes a compact topological space.[♭] Moreover the cylinder sets, being the balls of some radius, are both open and closed: clopen. It follows by definition that Ω has a basis of clopen sets and that every open set is a countable union of cylinder sets (there being only a countable number of cylinder sets because the finite causets are only countable in number). Since each element of the event algebra $\mathfrak{S}$ is itself a (finite) union of cylinder sets, we can conclude that the open sets are precisely the members of $\bigvee\mathfrak{S}$, the closed sets, their complements, being then the members of $\bigwedge\mathfrak{S}$. The events that belong to both these families are the clopen events, and they clearly include all of $\mathfrak{S}$, because a finite union of open [respectively closed] sets is also open [closed]. Let us prove the converse, that every clopen event belongs to $\mathfrak{S}$.

Lemma (4.1) $\quad \bigvee\mathfrak{S} \,\cap\, \bigwedge\mathfrak{S} \;=\; \mathfrak{S}$

Proof We are asked to prove that $\mathfrak{S}$ comprises precisely the clopen subsets of Ω. Since we already know that every $A \in \mathfrak{S}$ is clopen, it suffices to verify that any clopen A, i.e. any $A \in \bigvee\mathfrak{S} \cap \bigwedge\mathfrak{S}$, also belongs to $\mathfrak{S}$. Let $A \in \bigvee\mathfrak{S}$. By definition A is a union of cylinder sets: $A = Z_1 \cup Z_2 \cup Z_3 \ldots$. Now this sequence either terminates at a finite stage or it does not. If it does terminate then A is a finite union of cylinder sets, whence a member of $\mathfrak{S}$, and we are done. If it does not terminate, then we can find a sequence of points $x_j \in A$ which escapes from every Z_j; and because Ω is compact, we can suppose that this sequence converges to some $x \in \Omega$. This x cannot lie in any given Z_k because it is a limit of points x_j which eventually belong to the closed set $\Omega \backslash Z_k$; consequently $x \notin A = \cup_k Z_k$. We have thus constructed a sequence of points of A which converge to a point outside A, meaning that A is not closed. It thus cannot be clopen. Or if it is clopen, then we are back to the terminating sequence and the conclusion that $A \in \mathfrak{S}$. $\square$

Turning to the 2-site hopper, we need to make only one change to what has been written above for causets. The histories are now sequences of digits, 0 or 1, beginning with 0, and the integer n that occurs in the definition (5) is now the largest index such

[♭] Proven explicitly in the next sub-section.

that the two subsequences $(0\ a_1\ a_2\ \cdots\ a_n)$ and $(0\ b_1\ b_2\ \cdots\ b_n)$ coincide. The rest is all the same. The history space Ω is still a compact metric space, the cylinder sets are clopen and generate the topology, etc.

the tree of truncated histories

We have already seen in section 2, that a point of Ω, that is to say a history, can be construed as a path γ through the tree $\mathfrak{T}$, each node of which is a "truncated history", meaning, as the case may be, either a finite causet or a finite sequence of binary digits.* Flowing from this correspondence between histories and paths through $\mathfrak{T}$ is a different way to characterize certain types of events, including the open sets $\bigvee \mathfrak{S}$ and more generally the events in $\mathfrak{R} \bigvee \mathfrak{S}$.

Consider first a cylinder set $Z = \mathrm{cyl}(h)$, where $h \in \Omega(n)$ is a truncated history. Which paths γ correspond to this cylinder set? By definition they are just the paths whose corresponding histories reproduce h when truncated at the n^{th} stage, that is, they are precisely the paths that *pass through the node* in $\mathfrak{T}$ that represents h, which I will denote either as h itself or as $\mathrm{node}(h)$ in order to emphasize that h is being treated as a node in $\mathfrak{T}$. Because $\mathfrak{T}$ is a tree, such a path necessarily follows one of the branches emanating from $\mathrm{node}(h)$; it then remains forever in the "upward subset" of $\mathfrak{T}$ consisting of all descendants (in $\mathfrak{T}$) of h. In this way, every open event $A \subseteq \Omega$ can be represented by an upward-closed subset $\alpha \subseteq \mathfrak{T}$, and vice versa, given such a subset the paths that enter (and consequently remain in) α comprise an open event $A \subseteq \Omega$. More generally, every subset $\alpha \subseteq \mathfrak{T}$ gives rise to an event $S(\alpha)$ by the same rule:

DEFINITION $\quad S(\alpha) = \{\gamma : \gamma \text{ is eventually in } \alpha\}$

Here, γ is a point of Ω, represented as a path $\gamma = (h_0, h_1, h_2 \cdots)$ through $\mathfrak{T}$, and the statement that this path is *eventually* in α means that $(\exists n_0)(\forall n > n_0)(h_n \in \alpha)$. Evidently, $S(\cdot)$ commutes with the Boolean operations: $S(\alpha\beta) = S(\alpha)S(\beta)$, $S(\alpha \backslash \beta) = S(\alpha) \backslash S(\beta)$, $S(\alpha + \beta) = S(\alpha) + S(\beta)$, etc. (where $\alpha + \beta := (\alpha \cup \beta) \backslash (\alpha\beta)$ is the Boolean operation of "addition modulo 2").

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* The paths under consideration in what follows will usually begin at the "root" of $\mathfrak{T}$ (corresponding to the cylinder-set Ω), but sometimes they will originate at some other node of $\mathfrak{T}$. The two cases are actually interchangeable because any path not originating at the root has a unique extension back to it, $\mathfrak{T}$ being a tree.

As a further aid to intuition, one can conceive of certain types of events in terms of "properties" acquired or lost in the course of the process under consideration. Formally, this corresponds closely to the characterization by sets of nodes in $\mathfrak{T}$, but it carries perhaps a more "evolutionary" feeling. For example consider the event of return analyzed earlier. One can cook up a "property" which the particle possesses when, and only when, it has returned to the origin. By definition, this property of "having returned" is *hereditary* in the sense that, once acquired, it can never be lost. Topologically, the set of all paths γ which acquire a hereditary property yields an open subset of Ω, as is easily corroborated if one thinks through the definitions. Dually, a property that once lost can never be regained, but that every path begins with, corresponds to a closed set (causet example: being an originary). And a property that can be acquired but never regained if lost yields an event of the form $A \backslash B$, where both A and B are open. (Hopper example: visiting $x = 1$ but not $x = 2$. Notice that this third type of property includes both of the previous two as special cases.) In terms of sets of nodes like the sets α discussed above, the first type of property is an upward-closed subset of $\mathfrak{T}$, the second is a downward-closed subset, and the third is a *convex* subset, defined as a subset of $\mathfrak{T}$ that contains, together with nodes h_1 and h_2, every node that lies on some path from h_1 to h_2. In order-theoretic language for the poset $\mathfrak{T}$, this just says that α includes the *order-interval* between any two of its elements.[†] In Sections 5 and 6, the events of the form $S(\alpha)$ for some convex α will be among those for which we will able to produce a canonical representation as a limit of clopen events.

As an application of some of these ideas, let us prove the assertion made earlier that Ω is topologically compact. By a standard criterion for compactness, it suffices to prove that any covering of Ω by cylinder sets has a finite sub-covering, so consider an arbitrary collection of cylinder sets $Z \in \mathfrak{Z}$ that covers Ω. In relation to $\mathfrak{T}$, such a covering is a collection of nodes which no path γ can avoid forever. Now the (incomplete) paths that do avoid these nodes fill out a subtree[♭] $\mathfrak{T}'$ of $\mathfrak{T}$, with the property that no path γ can remain within $\mathfrak{T}'$ forever. But it is well-known that such a tree can have only a finite number of nodes, assuming that no node has an infinite branching number. (This has been called the "infinity lemma" of graph theory.) The maximal elements of $\mathfrak{T}'$ thus furnish a finite

[†] The order-interval delimited by elements x and y of some poset is $\{z : x \prec z \prec y\}$.

[♭] a downward-closed subset of $\mathfrak{T}$.

collection of nodes that every path must encounter. In their guise as cylinder-sets these nodes constitute then a finite subcover of Ω.

5. Set-theoretic limits of events

The most elementary kind of limit that one can imagine for a sequence of events partakes of neither metric nor topology nor measure; it is purely set-theoretic. We have already seen how increasing sequences of clopen sets yield the open sets $\bigvee\mathfrak{S}$, while decreasing sequences of clopen sets yield the closed sets $\bigwedge\mathfrak{S}$. In both cases the operative concept of limit emerges more or less automatically. Going beyond these two types of approximation, we can recognize a more general concept of which $\bigvee$ and $\bigwedge$ are special cases. Let X_j be a sequence of subsets of Ω, and deem it to be convergent when we have for any point x of Ω that eventually $x \in X_j$ or eventually $x \notin X_j$. In such a case, we will write $X = \lim X_j$, where of course X consists of those x that realize the first alternative of being eventually in X_j . The set of all events obtainable in this way as limits of events $A_j \in \mathfrak{S}$, I will denote as $\mathrm{Lim}\,\mathfrak{S}$. Notice that 'lim' commutes with the Boolean operations:

$$\lim(A_n \cup B_n) = (\lim A_n) \cup (\lim B_n) \ , \quad \text{etc.} \tag{6}$$

In trying to extend our vector-measure μ beyond the clopen events, one might hope that one could at least get as far as $\mathrm{Lim}\,\mathfrak{S}$. Were μ an ordinary measure, this would be true, because $\lim A_j$ would be sandwiched between the measurable sets $\limsup A_j = \cap_j \cup_{k>j} A_k$ and $\liminf A_j = \cup_j \cap_{k>j} A_k$, both of which are equal to $\lim A_j$ when the latter exists. This would ensure that $\lim A_j$ was measurable and that $\lim \mu(A_j) = \mu(\lim A_j)$. But with quantal measures this argument is not available, and it turns out that convergence can fail already for certain decreasing sequences of clopen events whose measures diverge [14] to infinity. On the other hand, convergence succeeds for many other sequences, and one might hope that the failures were confined to physically uninteresting questions.

Of course, the failure of convergence in even some cases is likely to contaminate other cases, making it dubious that $\mu(A)$ can be defined without some further limitation on the sequence A_j beyond the mere requirement that $\lim A_j = A$. In the next two sections we will investigate some restrictions of this sort. For now, let's notice that for open sets A there exists a very naturally defined canonical sequence of events $A_n \in \mathfrak{S}_n$ converging to A. Namely, we can take for A_n the union of all the cylinder sets from $\mathfrak{Z}_n$ that are contained within A. This yields a "best approximation to A at stage n" in the sense that A_n couldn't be enlarged without the sequence losing its increasing nature.

Dually, one immediately obtains a canonical choice of sequence for any closed event B (just apply complementation to the sequence of clopen events approximating $\Omega\backslash B$), but having thus two different classes of canonical sequences introduces an ambiguity for events that are both open and closed. Fortunately, the ambiguity in this case does no harm because a clopen event necessarily belongs to $\mathfrak{S}$, according to Lemma 4.1. Both the increasing and decreasing canonical sequences thus terminate at a finite stage: they differ only transiently.

Leaving aside questions of uniqueness and of convergence of measure, one might ask how many events the above limit process can access, even in the best case. That is, how many of the interesting questions even belong to $\operatorname{Lim}\mathfrak{S}$ at all? With reference to the causal set case, recall first of all that one encounters all the stem-events without ever leaving the open sets $\bigvee\mathfrak{S}$. Remembering also that $\operatorname{Lim}\mathfrak{S}$ is closed under the Boolean operations, we can thus say on the positive side that every finite logical combination of stem-events is available within $\operatorname{Lim}\mathfrak{S}$ (as also the entire event-algebra $\mathfrak{R}\bigvee\mathfrak{S}$ of course). On the negative side however, we can notice that events like the post-event and (for the particle case) the event of infinite return fall outside of $\operatorname{Lim}\mathfrak{S}$, as a consequence of the following lemma.

Lemma (5.1) Let $A \subseteq \Omega$. If both A and $\Omega\backslash A$ are dense subsets of Ω then $A \notin \operatorname{Lim}\mathfrak{S}$.

Proof In the following $A \perp B$ will mean that A and B are disjoint. Suppose, for contradiction, that $A = \lim A_n$ with $A_n \in \mathfrak{S}$, and write $\overline{A}_n$ for the complement $\Omega\backslash A_n$, also taking note of the fact that $\overline{A}_n$, like A_n itself, is clopen. We will find inductively a subsequence $A_{n_1}, A_{n_2}, A_{n_3}, \cdots$ of the A_n and a matched sequence of clopen sets $B_1 \supseteq B_2 \supseteq B_3 \cdots$ such that B_j is alternately included in and disjoint from A_{n_j}.

step 1. To start with, put $n_1 = 1$ and $B_1 = A_{n_1}$. We have $B_1 \subseteq A_1$.

step 2. Next observe that since B_1 is open and $\Omega\backslash A$ is dense, there exists $x \in B_1 \cap (\Omega\backslash A)$. Then since $x \notin A = \lim_n A_n$, there exists by hypothesis some $n_2 > n_1$ such that $x \notin A_{n_2}$, i.e. $x \in \overline{A}_{n_2}$. Put $B_2 = \overline{A}_{n_2} \cap B_1$, which is again clopen since both $\overline{A}_{n_2}$ and B_1 are clopen. We have $B_2 \subseteq B_1$ with $B_2 \perp A_{n_2}$.

step 3. For step 3, we proceed exactly as in step 2 with the roles of A and $\Omega\backslash A$ interchanged. Namely we observe that since B_2 is open and A is dense, there exists $x \in B_2 \cap A$. Then since $x \in A = \lim_n A_n$, there exists by hypothesis some $n_3 > n_2$ such that $x \in A_{n_3}$.

Put $B_3 = A_{n_3} \cap B_2$, which is again clopen since both A_{n_3} and B_2 are clopen. We have $B_3 \subseteq B_2 \subseteq B_1$ with $B_3 \subseteq A_{n_3}$.

Now proceed inductively to produce $B_4 \subseteq A_{n_4}$, $B_5 \perp A_{n_5}$, etc. Finally put $B = \lim_n B_n = \bigcap_{n=1}^{\infty} B_n$ and note that B is non-empty since the B_n are all compact. (Indeed, every event in $\mathfrak{S}$ is compact, being a closed subset of the compact space Ω.) Pick any $x \in B$. For odd j we have $x \in B_j \subseteq A_{n_j} \Rightarrow x \in A_{n_j}$. For even j we have $x \in B_j \perp A_{n_j} \Rightarrow x \notin A_{n_j}$. Thus the A_n vacillate between including and excluding x, contradicting our assumption that $\lim A_n$ exists. $\square$

The lemma applies to the post-event because, no matter how far the growth process has proceeded, the growing causet "still has a free choice" whether to end up with a post or without one (and exactly the same thing can be said for the event of infinite return). But this freedom means precisely that both the post-event and its complement are dense in Ω.

Lemma 5.1 shows that $\operatorname{Lim}\mathfrak{S}$ is far from containing every event of potential interest, but one might wonder exactly how far. One answer comes from Exercise (22.17) of reference [12], according to which $\operatorname{Lim}\mathfrak{S}$ equals what is called Δ^0_2, defined to be the intersection of $\bigvee\bigwedge\mathfrak{S}$ and $\bigwedge\bigvee\mathfrak{S}$. This places $\operatorname{Lim}\mathfrak{S}$ at a very low level of the so called "Borel hierarchy", which continues on for $\aleph_1$ steps beyond Δ^0_2 before it exhausts the Borel subsets of Ω. In this sense the limiting process 'lim' does not take us very far beyond the clopen events. On the other hand, we have also seen that by applying 'lim' more than once, one can reach, for example, the post-event. How many events can one reach in this manner? When combined with other results in [12], (22.17) therein also answers this question by implying that (transfinite but still countable) iteration of the 'lim' operation suffices to produce any Borel set. In this sense the lim operation is quite far reaching, given that an event of interest but not falling within the Borel domain would be hard to conceive of.

6. Canonical approximations for certain events

For an event $A \subseteq \Omega$ which is open with respect to the topology defined in Section 4, that is for $A \in \bigvee\mathfrak{S}$, we have already discovered one canonical sequence A_n of approximations to A. The cylinder sets $\mathfrak{Z}_n$ "at stage n" provide a kind of "mesh" in Ω whose fineness increases with n, and our canonical choice of approximating event at stage n was

$$A_n = \bigcup \{Z \in \mathfrak{Z}_n \ : \ Z \subseteq A\} \ , \tag{7}$$

the biggest member of $\mathfrak{S}_n = \mathfrak{R}\mathfrak{Z}_n$ which can fit inside A. As we have seen, the A_n converge to A in the sense defined in section 5, but of course there exist many other sequences $B_n \in \mathfrak{S}_n$ which also converge to A in this sense, and when the vector-measure μ is not of bounded variation, there is no guarantee that the corresponding sequences $\mu(A_n)$ and $\mu(B_n)$, if they converge at all, will converge to the same limit. In general they doubtless will not if B_n is chosen with sufficient malice. In the face of such ambiguity, one might still hope to find some reasonably inclusive event-algebra $\mathfrak{A} \supseteq \mathfrak{S}$ and for each event $A \in \mathfrak{A}$ a canonical approximating sequence of events $A_n \in \mathfrak{S}_n$ with $\lim A_n = A$ and such that $\mu(A_n)$ was a convergent sequence in Hilbert space. The vector $\lim_n \mu(A_n)$ could then be adopted as the definition of $\mu(A)$.

One snag that this perspective encounters is apparent already for the case where we are approximating open sets A and B, and our canonical approximations A_n and B_n are the ones given by (7). From $\lim A_n = A$ and $\lim B_n = B$ it does indeed follow, as we have already noted, that $\lim(A_n \cap B_n) = A \cap B$ and $\lim(A_n \cup B_n) = A \cup B$. For the case of intersection it even follows that the events $(A_n \cap B_n)$ provide the canonical approximations to the open event $A \cap B$, but the analogous conclusion fails for the case of union because the canonical approximation $(A \cup B)_n$ will in general be larger than $(A_n \cup B_n)$, since some cylinder set $Z \in \mathfrak{Z}_n$ can, by "straddling the boundary" between A and B, be included in $A \cup B$ without being included in either A or B. One would thus obtain different approximating sequences for $A \cup B$, depending on whether one regarded it as an open set in its own right or as the result of uniting A with B. In the next section we will begin to see what it would take to render this kind of ambiguity harmless. For now however, let's ignore that issue and consider simply the question of finding unambiguous approximating sequences for as many members of $\mathfrak{R} \bigvee \mathfrak{S}$ $(= \mathfrak{R} \bigwedge \mathfrak{S})$ as possible.

To that end, let's return to the tree $\mathfrak{T}$ of truncated histories and the method of representing certain events by subsets $\alpha \subseteq \mathfrak{T}$. Although we didn't make it explicit earlier, it is clear that a sequence of events $A_n \in \mathfrak{S}_n$ is equivalent to a set of nodes $\alpha \subseteq \mathfrak{T}$. Indeed, each A_n is a union of cylinder sets $Z \in \mathfrak{Z}_n$, and each such cylinder set corresponds to a node in $\mathfrak{T}_n$, the n^{th} level of $\mathfrak{T}$. This associates to each A_n a set of nodes at level n, and amalgamating the nodes of all levels into a single collection yields α. Conversely, given $\alpha \subseteq \mathfrak{T}$ we obtain A_n as the union of the cylinder sets that correspond to the α-nodes at level n. Since the correspondences between cylinder sets $Z \in \mathfrak{Z}_n$, nodes in $\mathfrak{T}_n$, and truncated histories $\gamma \in \Omega(n)$ are so close, I will often identify all three with one another, speaking for example of a cylinder set Z as a node in $\mathfrak{T}$. When this is done, we can express the correspondence between node-sets α and approximating sequences (A_n) in a simple formula by writing $A_n = \bigcup(\alpha \cap \mathfrak{Z}_n)$.

Now let α be any set of nodes and let the A_n be the corresponding sequence of events. Recall that we defined $S(\alpha)$ as the event that γ is *eventually* in α:

$$S(\alpha) = \{\gamma \,:\, (\exists N)(\forall n > N)(\gamma_n \in \alpha)\} \ .$$

Dually one can also define $\widetilde{S}(\alpha)$ as the event that γ is *repeatedly* in α:

$$\widetilde{S}(\alpha) = \{\gamma \,:\, (\forall N)(\exists n > N)(\gamma_n \in \alpha)\} \ .$$

It follows, simply by tracing through the definitions, that

$$\begin{aligned} S(\alpha) &= \liminf A_n \ , \\ \widetilde{S}(\alpha) &= \limsup A_n \ . \end{aligned} \tag{8}$$

Since $\lim A_n$ exists if and only if $\liminf A_n = \limsup A_n$ (in which case their common value equals $\lim A_n$), we learn that the events of the form $\lim A_n$ are precisely those for which $S(\alpha) = \widetilde{S}(\alpha)$, which in turn are precisely those such that no path γ can leave and re-enter α more than a finite number of times. Evidently this property generalizes the concept of convexity which we met with earlier. Notice incidentally that equations (8) imply that the forms $S(\alpha)$ and $\widetilde{S}(\alpha)$ don't reach beyond $\bigvee\bigwedge\mathfrak{S}$ and $\bigwedge\bigvee\mathfrak{S}$, known in descriptive set theory as Σ^0_2 and Π^0_2, respectively. Roughly, they reach as far as events whose complexity is that of the post event. Very optimistically, one might hope to go beyond this and find for any Borel set $A \subseteq \Omega$, some sort of canonical presentation in terms of clopen events, but in this section we will not venture outside of $\mathfrak{R}\bigvee\mathfrak{S}$, the finite Boolean combinations of opens. Since $\mathfrak{R}\bigvee\mathfrak{S} \subseteq \operatorname{Lim}\mathfrak{S}$, all such events can be expressed as $S(\alpha)$ for some subset $\alpha \subseteq \mathfrak{T}$.

What we are asking for is a sort of "normal form" for events E in $\mathfrak{R}\bigvee\mathfrak{S}$. As a first step in that direction, let us prove that every such event can be expressed as a disjoint union of events, each of which has the form, open\open, or equivalently, open $\cap$ closed.

Lemma (6.1) Let $E \in \mathfrak{R}\bigvee\mathfrak{S}$ be a finite logical combination of open events. Then there exists a decreasing sequence of open events $E^1 \supseteq E^2 \supseteq E^3 \cdots \supseteq E^K$ such that $E = E^1 + E^2 + E^3 \cdots + E^K = E^1 \backslash E^2 \sqcup E^3 \backslash E^4 \sqcup \cdots$, where '$\sqcup$' denotes disjoint union.

Moreover the E^j are formed from the original events using only the operations of union and intersection.

PROOF In this proof, as in the statement of the lemma, we use the operation of Boolean addition,

$$A + B = (A \cup B) \backslash (A \cap B) \ , \tag{9}$$

and we write the intersection of two sets as their product. Any Boolean combination of sets is then a polynomial in these sets, and since products of open sets are open, any Boolean combination of open events can be expressed simply as a Boolean sum of open events. Given these facts, a proof by induction is not hard to devise, but it seems clearer just to illustrate the pattern involved with the cases of $K = 2, 3$. For two events we have $A + B = (A + B + AB) + AB = A \cup B + AB$. For three we have $A + B + C = A + (B + C) = A + (B \cup C + BC) = (A + B \cup C) + BC = (A \cup B \cup C + A(B \cup C)) + BC = A \cup B \cup C + (A(B \cup C) + BC) = A \cup B \cup C + A(B \cup C) \cup BC + A(B \cup C)BC = A \cup B \cup C + (AB \cup AC \cup BC) + ABC$. The "inclusion-exclusion" pattern that is evident here emerges with particular clarity when one interprets Boolean addition as addition of characteristic functions modulo 2. The final equation in the statement of the lemma then follows directly from the fact that the E^j are decreasing. One can also restate the essence of the proof in a simple formula: $\sum_{j=1}^K A_j = \sum_{j=1}^K B_j$, where $B_j = \{x : x \text{ belongs to at least } j \text{ of the } A_k\}$, this being clearly a union of intersections of the A_k. $\square$

Given any set E expressed as in the lemma, we get immediately the approximations $E_n = E_n^1 + E_n^2 + \cdots + E_n^K$, where E_n^j is our canonical n^{th} approximation to the open set E^j, and thence the corresponding sets of nodes $\alpha_n = \alpha_n^1 + \alpha_n^2 + \cdots + \alpha_n^K$ together with their union

$$\alpha = \cup_n \alpha_n \tag{10}$$

However, this construction is only a first step toward uniqueness, because the resulting α still depends on the original choice of the E^j, which are not given to us uniquely by the lemma.

In working toward a unique approximating sequence, let us concentrate on the simplest case of an event $E = A \backslash B$ which is the difference of only two open sets $B \subseteq A$ (corresponding to $K = 2$ in the lemma). Can we render A and B unique in this case? It's not difficult to demonstrate that if we gather together all pairs $A \supseteq B$ such that $E = A \backslash B$, then the union of all the sets A and the union of all the sets B yields another such pair.

Evidently this "biggest pair" is unique and uniquely determined by the original event E. This in turn yields [by (6)] a canonical sequence of approximations E_n to E of the form, $E_n = A_n \backslash B_n$, where $A_n \in \mathfrak{S}_n$ and $B_n \in \mathfrak{S}_n$ are the canonical n^{th} approximations to A and B. In terms of the equivalent node-sets α_n and β_n, these approximations are given by $\alpha_n \backslash \beta_n$, whose union over n corresponds to the node-set α of equation (10).

Although the node-set that we have just found is canonical and concisely defined, one might wish for a more constructive route to it, or at least a characterization of it in terms of more easily verifiable necessary and sufficient conditions. The remainder of the present section will develop a prescription of this sort. In fact, I am not certain that the second prescription will be strictly equivalent to the first. If it is, that is all to the good since we will then not be forced to choose between the two. If it isn't, that doesn't really matter, since the second prescription stands on its own and, being more concrete, is likely to be more useful in practice.

Lemma (6.2) If α and β are upward-closed subsets of $\mathfrak{T}$ with $\alpha \supseteq \beta$ then $\alpha \backslash \beta$ is convex.

Proof We are to show that no path between two nodes x and y in $\alpha \backslash \beta$ can contain nodes outside of $\alpha \backslash \beta$. Equivalently, no path which has left $\alpha \backslash \beta$ can ever re-enter it. But since α is upward-closed no path from $x \in \alpha$ can leave α, therefore it can leave $\alpha \backslash \beta$ only by entering β; it then must remain in β (which is also upward-closed) forever, and consequently can never re-enter $\alpha \backslash \beta$. $\square$

Now let $E = A \backslash B$ with A, B open and let α and β be the corresponding node-sets. Since A and B are open, both α and β are upward-closed subsets of $\mathfrak{T}$. We also know that $A = S(\alpha)$, $B = S(\beta)$ and $A \backslash B = S(\alpha \backslash \beta)$. The lemma then teaches us that $A \backslash B = S(\widehat{\alpha})$, with $\widehat{\alpha}$ a convex subset of $\mathfrak{T}$. The converse is true as well:

Lemma (6.3) If $\widehat{\alpha} \subseteq \mathfrak{T}$ is convex then $S(\widehat{\alpha}) = A \backslash B$ for some open sets A and B.

Proof Recalling that we have identified points of Ω with infinite paths γ through $\mathfrak{T}$, let A be the set of all paths that enter $\widehat{\alpha}$, and let B be the subset of these that subsequently leave $\widehat{\alpha}$. By definition $S(\widehat{\alpha}) = A \backslash B$, but both A and B are open because the property of "having entered $\widehat{\alpha}$" and the property of "having left $\widehat{\alpha}$" are both hereditary. $\square$

Henceforth, we will just deal with the convex subset $\widehat{\alpha}$, renaming it to plain α for simplicity. That is, we will be concerned with a fixed event E of the form (open\open) and with a convex set of nodes $\alpha \subseteq \mathfrak{T}$ such that* $E = S(\alpha)$.

Let us say that $\alpha \subseteq \mathfrak{T}$ is *prolific* if it lacks maximal elements. A second requirement that adds itself very naturally to convexity is the condition that α be prolific in this sense. Given convexity, this is equivalent to saying that every node $x \in \alpha$ originates a path that remains forever within α. In the opposite case, α will contain "sterile" nodes from which all paths eventually leave α for good. It is clear that removing these sterile nodes will not alter E, nor will it spoil the convexity of α. We can therefore always arrange that α be both convex and prolific. The "pruning" of the "sterile" nodes in order to render α prolific also appears as a very natural operation when it is expressed in terms of cylinder sets Z. It simply removes from α those Z which are disjoint from E.

We have now arranged for α to be convex and prolific, but this does not yet make it unique, since for example we could remove all the nodes up to any fixed finite level n without altering $S(\alpha)$. If we did so, however, we might create a situation where, for example, some cylinder set Z was wholly included in E without Z itself (regarded as a node in $\mathfrak{T}$) belonging to α. To remedy this kind of lacuna, we can adjoin to α every node Z such that every path originating at Z eventually enters α. It is again easy to see that adjoining these nodes will not interfere with α being convex and prolific.

In this last step we have, in a manner of speaking, completed α toward the past, but in fact there is cause to carry this process of "past-completion" farther by adjoining still other nodes to α. These additional nodes are perhaps not such obvious candidates as the previous ones, but throwing them in as well (which I think corresponds to enlarging the open set A) will provide us with the uniqueness we are seeking.

Definition Let $x \in \mathfrak{T}$ and $\alpha \subseteq \mathfrak{T}$. Then $x \prec \alpha$ means that x precedes some node in α: $(\exists y \in \alpha)(x \prec y)$.

Remark In terms of cylinder sets, $Z_1 \prec Z_2 \iff Z_2 \supseteq Z_1$.

Definition The *exclusive past* of α is the set of nodes strictly below α: $\{x \notin \alpha \,:\, x \prec \alpha\}$.

Using this definition, let us say that α is *past-complete* if its exclusive past P is prolific, which in turn says that any node in P originates a path that repeatedly visits P. I claim we can render α past-complete be adjoining to it all nodes below α that fail to satisfy this

* In view of (8) we would also want in general to require that $S(\alpha) = \widetilde{S}(\alpha)$, but this holds automatically when α is convex.

last condition, and furthermore that the resulting set of nodes α' will yield the same event E as α and will be convex and prolific if α itself was.

LEMMA (6.4) Let $\alpha \subseteq \mathfrak{T}$ be any set of nodes and let α' be its "past-completion" as just described. Then α' is past-complete. Moreover $S(\alpha') = S(\alpha)$ and $\widetilde{S}(\alpha') = \widetilde{S}(\alpha)$.

PROOF That $S(\alpha') \supseteq S(\alpha)$ is obvious. To prove that they are equal it suffices to show that no path can eventually remain within α' without also remaining eventually within α. Suppose the contrary, and let $\gamma \in S(\alpha')$ be a path which is repeatedly outside α. By passing to a tail of γ we can suppose that it is always within α'. Let $x \in \gamma$ be a node which is not in α and let $y \in \gamma$ be a later node of the same type. Since $y \in \alpha' \backslash \alpha$ it is by definition in P, the exclusive past of α. Hence x originates a path (namely γ) which visits P at y; and since there are an infinite number of nodes like y, γ visits P repeatedly. But this contradicts the criterion for having included x in α' in the first place.

The proof that $\widetilde{S}(\alpha') = \widetilde{S}(\alpha)$ is similar. It suffices to show that every path in $\widetilde{S}(\alpha')$ visits α repeatedly. Suppose the contrary, and let $\gamma \in \widetilde{S}(\alpha')$ be a path which is eventually outside α. By passing to a tail we can suppose that γ is always outside of α. Let $x \in \gamma$ be a node which is in α' and let $y_1, y_2, \cdots$ be a sequence of later nodes of γ which are also in α'. Since the y_j belong to $\alpha' \backslash \alpha$ they are by definition in P, the exclusive past of α. Hence x originates a path which returns repeatedly to P, contradicting the criterion for having included x in α' in the first place.

To complete the proof, we need to show that α' is past-complete. To that end let x be in the exclusive past of α'. Since any y in α' is either in α or in its exclusive past, and since $\alpha' \supseteq \alpha$, x is also in the exclusive past of α. Consequently, since x was not put into α', it originates a path γ that repeatedly visits the exclusive past of α. But by definition, no node in such a path would have been put into α' either, whence γ repeatedly visits the exclusive past of α', as was to be shown. $\square$

We also wish to prove that past-completion preserves the attributes of being prolific and convex. The first is easy because past-completion only adds nodes which are below some element of α, and this can introduce no new maximal element.

For the second, we need to demonstrate† that if α is convex, and if $x \prec y$ are nodes

† The demonstration that follows seems somehow longer than it ought to be. Intuitively it suffices to observe first that α' is built up from α by successive adjunction of maximal elements of its exclusive past, and second that adjoining such an element cannot spoil convexity.

in α' then the order-interval I delimited by x and y is also within α'. When $x \in \alpha$ the proof is simple, since y is either within α itself or precedes some element z which is. In either case the interval I is included in some second interval I' (possibly the same as I) with endpoints in α. Then $I' \subseteq \alpha$ because α is convex, whence also $I \subseteq I' \subseteq \alpha \subseteq \alpha'$, as desired. The remaining possibility is that $x \in \alpha' \backslash \alpha$, in which case it seems more convenient to deal with paths rather than intervals. From the definition of convexity, proving that α' is convex amounts to showing that no path originating from x can leave α' and then re-enter it. First, observe that since every node of α' precedes some node of α, no node of $\alpha' \backslash \alpha$ can follow a node of α; for if it did, it would also lie within the convex set α. In consequence, any path that exits α permanently exits α' as well. Now let γ be any path originating from $x \in \alpha' \backslash \alpha$, and as before write P for the exclusive past of α. By the definition of α', every path from x must eventually leave P. If it does so by leaving the past of α, $\{x \in \mathfrak{T} : x \prec \alpha\}$, then it certainly can never re-enter α'. If it does so by entering α, then it can exit α' only by exiting α, in which case it can never re-enter α or (as we just observed) α'.

So far, we have established the existence, for our event E, of a node-set α which is convex, prolific, and past-complete. Let us complete the story by proving that α is also unique. To that end, let α and β be two prolific, convex and past-complete node-sets such that $S(\alpha) = S(\beta)$. Does it follow that $\alpha = \beta$? In demonstrating that the answer is "yes", I will use the ad hoc notation $\mathbb{P}(\alpha)$ for the exclusive past of α as defined earlier, with $\overline{\mathbb{P}}(\alpha) = \alpha \cup \mathbb{P}(\alpha) = \{x \in \mathfrak{T} : (\exists y \in \alpha)(x \preceq y)\}$ being the *inclusive past*.

First, let us establish that α and β have equal inclusive pasts: $\overline{\mathbb{P}}\,\alpha = \overline{\mathbb{P}}\,\beta$. In fact if $x \in \overline{\mathbb{P}}\,\alpha$ then x originates a path γ that visits α. Since α is prolific this path can be arranged to visit α repeatedly, and since α is convex, such a path can never leave α. Hence $\gamma \in S(\alpha)$, implying in particular that γ visits β, whence $x \in \overline{\mathbb{P}}\,\beta$. The converse follows symmetrically.

Now suppose for contradiction that there exists $x \in \alpha \backslash \beta$. Such an x belongs by definition to $\overline{\mathbb{P}}\,\alpha$, hence to $\overline{\mathbb{P}}\,\beta$, hence to $\mathbb{P}\,\beta$, which in turn is prolific by definition of past-completeness. Thus x originates a path γ that repeatedly visits $\mathbb{P}\,\beta$. I claim that γ must leave α at some stage. (Otherwise $\gamma \in S(\alpha) \Rightarrow \gamma \in S(\beta) \Rightarrow \gamma$ eventually in β, whence γ could never again visit $\mathbb{P}\,\beta$.) And since α is convex, γ must remain outside of α once it has left. On the other hand, γ must continue to visit $\mathbb{P}\,\beta$, which in turn is a subset of $\overline{\mathbb{P}}\,\beta = \overline{\mathbb{P}}\,\alpha$. But if γ really visited some $y \in \overline{\mathbb{P}}\,\alpha$, then by definition we could divert it at y

to some other γ' that would re-enter α, something that we just proved to be impossible. This completes the proof of:

THEOREM (6.5) Every event E of the form $E = A \backslash B$ with A and B open can be expressed as $E = S(\alpha) = \widetilde{S}(\alpha)$ for a unique set of nodes α which is convex, prolific and past-complete.

The theorem furnishes a canonical set α of nodes corresponding to E, and as explained earlier, one obtains immediately from such an α a canonical sequence of approximants E_n to E such that $E = \lim_n E_n$. We have thus reached our immediate goal.

The canonical approximating sequences of the theorem provide a good reference point for further developments, and we have learned how to arrive at them step by step, starting from the open sets A and B. Nevertheless it seems unlikely that we can limit ourselves to these particular approximants in general. Rather, as remarked already at the beginning of this section, one will in general have to deal with many different sequences converging to the same event, unless perhaps it is possible to devise canonical sequences which are closed under the Boolean operations.

We already encountered an ambiguity of this nature when we noticed that our original, increasing canonical approximants (7) for open events (call them "C1") are not fully compatible with the Boolean operation of complementation. Specifically for a clopen event E, these C1 approximants depend on whether one derives them directly from E or by complementing the corresponding approximants for the open event $\Omega \backslash E$. We run into a further, but related conflict if we now compare the C1 approximants with those of the above theorem (call them "C2"). For an open event E the C1 approximant E_n is nothing but the biggest member of $\mathfrak{S}_n$ included within E. But if we view E as the difference $E = A \backslash B$, with A being E itself and $B = 0$ being the empty event, the theorem provides a different set of approximants E_n. In general, the two disagree, as one can appreciate if one notices that the pair $(E\ 0)$ is not the "biggest one" yielding E.

Consider for example the 2-site hopper event that the particle does not remain forever at its starting site 0, but that the first time it hops to site 1 it immediately returns to 0. This event is a union of cylinder sets corresponding to truncated trajectories of the shape $(0, 0, 0, \cdots 0, 1, 0, *)$ where the star '$*$' represents any finite sequence of zeros and ones. For this event, the node-set α_1 of type C1 consists of precisely the truncated trajectories just indicated. But that set of nodes is not past-complete. Its completion, the type C2 node-set α_2, contains in addition the truncated trajectories $(0, 0, 0, \cdots 0, 1)$. Notice that

α_2 differs from α_1 by an infinite number of nodes in this case. (Figure 4 illustrates this phenomenon.)

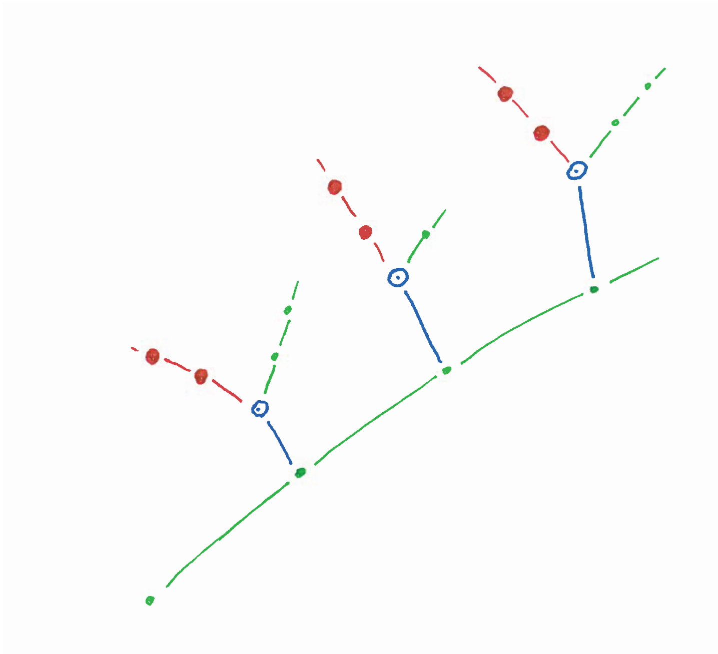

Figure 4. Illustrating past-completion in the tree shown. The nodes circled in blue "complete" those in solid red. Let the node-set be α_1 before completion and α_2 after completion. Evidentally $S(\alpha_1) = S(\alpha_2)$ but only α_1 is upward-closed.

We thus have to reckon with overlapping but in general incompatible prescriptions for different types of events. If one prescription were to be adopted exclusively, it should probably be C2, which covers more events than C1 does. (Incidentally, C2 resolves the aforementioned ambiguity in the C1 prescription in favor of treating clopen events as closed, not open.) On behalf of C1 one might make the counter-argument that monotonic convergence of the approximants is to be preferred, but this does not seem so compelling in the context of a *quantal* measure, which itself is not a monotonic set-function. Better than either choice, however, would be not having to choose at all because the alternative approximating sequences would all lead to the same extension of our initial quantal measure. The main thing for now is that we've discovered at least one canonical choice of clopen events E_n converging to any event of the form $E = A \backslash B$ with A and B open.

In the face of these various ambiguities it seems well to emphasize that none of them affect, in the causet case, the stem-events themselves, essentially because the latter are not only open but dense in Ω, or more physically because any growing causet that has not yet produced a given stem always retains a choice whether or not to do so. It follows that not only does the C1 prescription coincide with the C2 prescription for stem-events (its exclusive pasts being already prolific), but also the "biggest pair" prescription with which we began provably agrees with the C1 prescription. The same ought to apply to finite unions and intersections of stem-events, and similar comments could be made about the event of "return" in the hopper case.

Let me conclude this section by sketching very briefly how one might try to carry our successful "canonization" of $E = \text{open}\backslash\text{open}$ over to the general case where $E = E^1 + E^2 \cdots + E^K$, the E^j being open and nested. Just as earlier we found a "biggest pair", $A \supseteq B$, by forming unions of the individual events A and B, one can do the same thing here with the E^j to obtain (at least in principle) a canonical set of "biggest" open and nested events E^j such that $E = E^1 + E^2 \cdots + E^K$. Associating to each such E^j its canonical approximants E^j_n (in the C1 sense, say) then yields for E itself the approximants $E_n = E^1_n + E^2_n \cdots + E^K_n$ such that $\lim E_n = E$. In principle this achieves our goal, but it remains once again at a rather abstract level.

As before, we can attempt a more constructive development by working with the node-set $\alpha \subseteq \mathfrak{T}$ corresponding to our approximating sequence E_n, or perhaps with some similar node-set whose uniqueness can be established directly, and for which we can prove that $E = S(\alpha) = \widetilde{S}(\alpha)$. But how would our construction of α go in this more general case, and what would generalize the conditions that α be convex, prolific and past-complete? It is clear from Lemmas (6.2) and (6.3) that convexity is now too restrictive. In its place, one would probably put the more general requirement that no path γ could enter and leave α more than K times. Correspondingly one might then expect that α would decompose into subsets α^j that were convex in the strict sense. One might also try to arrange for each α^j to be prolific and past-complete, hoping that this would again confer uniqueness on the whole collection. If all this worked out, one would have constructed a canonical approximating sequence for any Boolean combination of open events, in particular for any Boolean combination of stem events.

A next step beyond $\mathfrak{R}\bigvee\mathfrak{S}$, if one could take it, would be to devise canonical approximations for larger families of events, starting with the collections $\bigvee\bigwedge\mathfrak{S}$ and $\bigwedge\bigvee\mathfrak{S}$ in which the post-event and its complement are to be found. An event A in either of these collections is accessible from $\mathfrak{S}$ as a limit of limits, but such a double limiting process can only make the potential ambiguities worse. For example, the event R^∞ of repeated return is in $\bigwedge\bigvee\mathfrak{S}$. It could be expressed as $\lim A_n$, where A_n = "returns at least n times", or it could be expressed instead as $\lim B_n$, where B_n = "returns at least once after $t = n$". Both A_n and B_n give decreasing sequences of open events and both converge to R^∞, but which sequence, if either, should be favored as canonical? Perhaps in certain cases, one could arrive at a canonical presentation by generalizing further our treatment above in terms of sets of nodes $\alpha \subseteq \mathfrak{T}$, but beyond this, it's not easy to guess how one might proceed. To devise canonical approximations for events of still greater complexity would seem to demand a fresh approach.

Finally, it might bear repeating here that uniqueness in and of itself does not guarantee compatibility with the Boolean operations. And I believe in fact that none of the prescriptions that this section has considered are compatible with the full set of such connectives, albeit some are compatible, for example, with complement or disjoint union (cf. figure 5). If there did exist a compatible prescription — or even a prescription compatible "modulo initial transients", which is just as good — that would weigh very heavily in its favor.

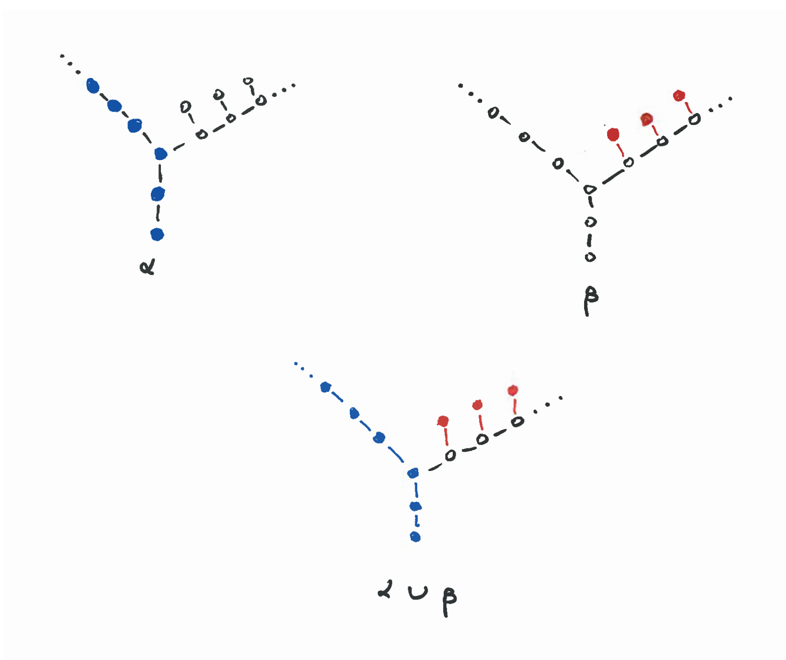

Figure 5. Two sets of nodes shown in red and blue. Both sets are convex, prolific and past-complete, but their union is not convex.

7. Evenly convergent sequences of events

We are given a vector-valued measure $\mu : \mathfrak{S} \to \mathfrak{H}$ defined initially on the finite unions of cylinder-events, and we wish to enlarge this initial domain $\mathfrak{S}$ so that it can embrace events like the stem-events and some of the other events we have been using as illustrations. In attempting such an extension or "prolongation" of μ, it is natural to think in terms of approximations, or more formally of limits. Let $A \subseteq \Omega$ be some event A outside the initial domain. In order to define $|A\rangle \equiv \mu(A)$ as a limit, one would aim to identify a sequence $A_n \in \mathfrak{S}_n$ of "best approximations to A" and one would then hope that the corresponding measures $|A_n\rangle \in \mathfrak{H}$ would also converge. If they did, then one would take their limit in $\mathfrak{H}$ to be the measure of A:

$$|A\rangle = \lim_{n \to \infty} |A_n\rangle \ .$$

Notice here that in attempting to define $|A\rangle = \mu(A)$ this way, we have relied on two independent notions of convergence, first the purely set-theoretic convergence of A_n to A in the sense of section 5 above, and second the topological convergence of the measures $|A_n\rangle$ to $|A\rangle$ in Hilbert space (say in the norm topology or perhaps the weak topology). One might question whether the first notion is really needed, given that the extension theorems of ordinary measure theory do without it, relying solely on the measure μ itself. Would it be possible to proceed similarly here? Unfortunately, this looks dubious, even though the vector $|A_n\rangle$ carries a certain amount of information about the event A_n (a very limited amount since, owing to quantal interference, very different events can share the same vector-measure.)

In the ordinary setting, where μ is real and positive, it defines a distance on the space of initially measurable sets modulo sets of measure zero, such that two events $A, B \in \mathfrak{S}$ are close when $\mu(A + B)$ is small. Extension of the measure then corresponds to completion of the metric space thereby defined [1]. Quantally, however, an event of small or zero measure is not negligible in the same way as it is classically, because of interference. Thus, if we tried to use the norm of $|A + B\rangle$ as a distance, it wouldn't even obey the triangle inequality. (Example: three disjoint events A, B, C as in the 3-slit experiment of [19] [20]; $|A+B\rangle = |B+C\rangle = 0$ but $|A+C\rangle \neq 0$.) Similarly trying to quotient the event-algebra by the events of measure zero would yield nonsense; it can even happen that all of Ω is covered by events of measure zero [17]. To establish an association between a vector $v = \lim |A_n\rangle$ and a definite event A in Ω, one thus seems to need an independent notion of convergence like that introduced above in section 5 and developed in section 6.

Accepting this apparent necessity, let us investigate how a limiting procedure might go in the important case of an open event $E \in \bigvee \mathfrak{S}$. In so doing, let us employ for E the canonical approximants $E_n \in \mathfrak{S}_n$ "of type C1", these being the simplest to work with and probably the first to suggest themselves for most people:

$$E_n = \bigcup \{ Z \in \mathfrak{Z}_n \; : \; Z \subseteq E \} \; . \qquad (11) = (7)$$

As we know, there is no guarantee in general that the corresponding vectors $|E_n\rangle$ will converge, but when they do, we'd like to regard E as "measurable" and to associate to it the measure $|E\rangle = \lim_n |E_n\rangle$. Below I will illustrate this procedure with the two-site and three-site hoppers, but first let us consider whether or not our criterion of convergence is adequate as it stands or whether it needs to be strengthened.

Recall in this connection that we had defined a second sequence of approximants for E "of type C2", related to the first ones by past-completion of the corresponding node-sets α. Would these approximants have led to the same set of measurable open events and to the same measures for them? A second question concerns compatibility with the Boolean operations, some form of which is needed if the extended measure is to be additive on disjoint events. Consider for example two disjoint open events A and B, and let $G = A + B = A \cup B$ be their their union, with G_n, A_n and B_n being the corresponding C1 approximants. If $|A_n\rangle + |B_n\rangle = |G_n\rangle$ held automatically it would follow immediately that $|A\rangle + |B\rangle = |G\rangle$, as desired. But plainly this is not automatic because $G = A \cup B$ can include cylinder sets Z that are not included separately in either A or B (see figure 6). We'd like the contribution from such Z to go away in the limit $n \to \infty$, and we'd also like any mismatch between our C1 and C2 sequences to go away. These two desiderata turn out to be closely related.

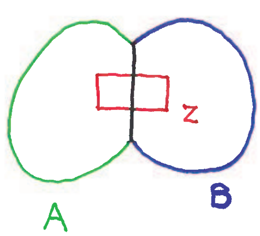

Figure 6. A cylinder set Z contributing to the difference (12).

Let us examine the difference,

$$|G_n\rangle - |A_n + B_n\rangle = |G_n\rangle - |A_n\rangle - |B_n\rangle \ , \tag{12}$$

more closely. In light of equation (11), this difference is just

$$\sum \{|Z\rangle \,:\, (Z \subseteq A+B)(Z \not\subseteq A)(Z \not\subseteq B)\} \ ,$$

but the Z here are not arbitrary cylinder sets. Rather, I claim that any Z which contributes to the above sum is special in that its overlaps with A and B are clopen (which implies that — within Z — the discrepancy disappears entirely in a later approximation). By the next lemma, something similar holds for the difference between the C1 and C2 approximants to any individual open set A.

Definition The cylinder set Z *straddles* the event A if it meets both A and its complement and if $Z \cap A$ is clopen. In symbols: $0 \neq ZA \neq Z$ and $ZA \in \mathfrak{S}$.

Remark Z straddles A iff it straddles the complement $\Omega \backslash A$

To see why the Z contributing to equation (12) are "straddlers" in this sense, it suffices to observe first that $ZA \equiv Z \cap A$ is open because both A and Z are open, and second that ZB is therefore closed, having the form closed\open: $ZB = Z \backslash ZA$. By symmetry both ZA and ZB are consequently both open and both closed.

Lemma Let A be an open event and let α be the corresponding node-set of type C1. If past-completion adds $Z \in \mathfrak{Z}$ to α then Z straddles A, and conversely.

Proof Since A is open, α is upward-closed, while for any α at all, it's true that $\beta = \{x \in \mathfrak{T} \,:\, x \prec \alpha\}$ is downward closed. Hence the difference $\beta \backslash \alpha$, the exclusive past of α, is also downward closed; i.e. it is a subtree of $\mathfrak{T}$. Now suppose without loss of generality that $Z \in \beta \backslash \alpha$. By definition, past-completion will adjoin Z to α iff no path originating at Z can remain within the subtree $\beta \backslash \alpha$. (It cannot leave $\beta \backslash \alpha$ and later return to it in this case because subtrees are convex.) But this means that the portion of $\beta \backslash \alpha$ above Z is actually finite (by the infinity lemma). Consequently, for n sufficiently big, every descendant of Z is either in α or below no node of α. Translated into the language of subsets of Ω, this says that at a sufficient degree of refinement n, every cylinder set $Z' \in \mathfrak{Z}_n$ and within Z is either fully included within A (the former alternative) or disjoint from A (the latter alternative). Now suppose that past-completing α does adjoin Z to it. Then ZA is the union of the Z' belonging to the first family and is therefore clopen, straddling A. Conversely, if ZA is clopen, then it is a union of cylinder sets $Z' \in \mathfrak{Z}_n$ for some n. $\square$

In view of these results, we can to some extent deal with the issues raised above by provisionally adding to our criterion of convergence the requirement that any "straddling" cylinder sets contribute negligibly in measure as $n \to \infty$.

Definition Let $A_n \in \mathfrak{S}_n$ be a sequence of clopen events. Call this sequence *evenly convergent* with respect to μ if the following hold:

(i) $A = \lim A_n$ for some $A \subseteq \Omega$

(ii) $|A\rangle = \lim |A_n\rangle$ for some $|A\rangle \in \mathfrak{H}$

(iii) $(\forall \varepsilon > 0)(\exists N)(\forall n > N)\ \sum || \, |Z\rangle \, || < \varepsilon$, where the sum ranges over all $Z \in \mathfrak{Z}_n$ that straddle A.

In view of the previous lemma, condition (iii) implies immediately that the C1 and C2 approximants for any open event E yield equivalent results. As desired, it also gives rise to additivity on disjoint open events, as we will see in the next theorem. In the statement of the theorem, the canonical approximants may for definiteness taken to be those "of type C1". As we have just seen, exchanging any of them for type C2 would have no effect. (Notice also in the statement of the theorem that the Boolean sum $A + B$ coincides with the union $A \cup B$ when A and B are disjoint: $AB = 0$.)

Theorem Let A and B be disjoint open events and let A_n and B_n be their canonical approximating sequences, with G_n being the canonical approximating sequence for $G = A + B$. If the first two sequences are evenly convergent then the third is also, and the measures add: $|A\rangle + |B\rangle = |G\rangle$.

Proof It will be convenient in the following to work with the canonical sequences given by (11), since for them the approximants to A and B will be disjoint. In the above definition of being evenly convergent , we need to establish conditions (i)–(iii) with A replaced by G. To begin with, condition (i), viz. $G = \lim G_n$, is true by construction.

Turning to condition (ii), we must verify that $|G\rangle = \lim |G_n\rangle$ with $|G\rangle = |A\rangle + |B\rangle$. We've already learned in connection with equation (12) that $|G_n\rangle - |A_n\rangle - |B_n\rangle$ is the sum of the measures $|Z\rangle$ of all those cylinder sets $Z \in \mathfrak{Z}_n$ that straddle both A and B. But this sum can be made arbitrarily small by choosing n big enough, since by hypothesis the sequence A_n is itself evenly convergent . (In more detail: $|| \, |G_n\rangle - |A_n\rangle - |B_n\rangle \, || = || \sum \{ |Z\rangle : Z \text{ straddles both } A \text{ and } B\} \, || \leq \sum \{ || \, |Z\rangle \, || : Z \text{ straddles both } A \text{ and } B\} \, || \leq \sum \{ || \, |Z\rangle \, || : Z \text{ straddles } A\} \, || \to 0$ as $n \to \infty$.) Therefore $\lim |G_n\rangle = \lim(|A_n\rangle + |B_n\rangle) = \lim |A_n\rangle + \lim |B_n\rangle = |A\rangle + |B\rangle$, as required.

Finally, we need to verify that the G_n themselves fulfill the third condition for being evenly convergent . To that end we will demonstrate that any cylinder event $Z \in \mathfrak{Z}_n$ that straddles G also straddles either A or B. The total norm of the straddlers of G will thus be bounded by the sum of the bounds for A and B, both of which go to zero as n goes to ∞; and therewith the proof will be complete. Suppose then that Z straddles $G = A + B = A \sqcup B$, where the symbol '$\sqcup$' denotes the union of disjoint sets. We have then $ZG = Z(A \sqcup B) = ZA \sqcup ZB$. By definition Z meets $A + B$, so suppose it meets A: $ZA \neq 0$. Now ZA is obviously open since both Z and A are open. It is also closed, being the difference of the clopen set $ZG = ZA \sqcup ZB$ and the open set ZB. Hence Z straddles A if it meets A at all, and in general it will straddle either A or B, as announced. $\square$

The theorem takes a first step toward arranging additivity of the extended measure, but of course disjoint open events is a special case. More generally, one would like to have similar theorems covering, say, arbitrary events in $\mathfrak{R} \bigvee \mathfrak{S}$ (not just open events) and arbitrary Boolean operations (not just disjoint union). For example, it's easy to establish for any open event E that $|E\rangle$ is defined iff $|\Omega \backslash E\rangle$ is defined, and that then $|\Omega \backslash E\rangle + |E\rangle = |\Omega\rangle$. To what extent such results can be obtained in general remains to be investigated.

Examples

Our friend, the return-event R, can serve to illustrate some of the definitions we have made. Let us start with the 2-site hopper, in which case $R' = \Omega \backslash R$, the event of "non-return", consists of the single history, $(0, 0, 0, \cdots)$. As we know, R itself is topologically open, and correspondingly $\Omega \backslash R$ is closed, as one can see directly from the fact that it is the limit of a decreasing sequence of clopen events of the form $R'_n = \text{cyl}(0, 0, 0, \cdots, 0)$, these being our canonical approximants for R'. In this case no cylinder event in $\mathfrak{Z}_n$ straddles R'_n since it itself is a cylinder event. To check that $|R'\rangle$ is well defined, then, we have only to check that the sequence $|R'_n\rangle$ converges. In fact, it converges trivially to 0, since $||\, |R'_n\rangle \,|| = (1/2)^{n/2}$. Thus, $|\Omega \backslash R\rangle = 0$ and non-return is *precluded* for the two-site hopper [14]. Taking complements shows then that $|R\rangle$ is defined and has the value $|R\rangle = |\Omega\rangle$.

In the context of the three-site hopper, the events of return and non-return become much more interesting. Classically, non-return "almost surely" does not occur in a finite lattice; its measure vanishes. Moreover, this conclusion obtains independently of what initial conditions one cares to assume. What we will find quantally? More generally, what will we find if, instead of asking whether the particle visits site 0, we ask whether it visits site 1 or 2? By symmetry, these questions become equivalent if we generalize our initial condition to admit different starting sites. Let us therefore consider (still more generally) an initial condition, in which each possible initial location contributes its own complex amplitude $\psi_0(j)$, $j = 0, 1, 2 \in \mathbb{Z}_3$. The measure of a cylinder-set of trajectories can then be derived from the 3-site analog of equation (2), generalized to allow for an arbitrary initial position x_0 , and with an additional factor of the initial amplitude $\psi_0(x_0)$ thrown in:

$$v_y = (U^{-n})_{yx_n} U_{x_n x_{n-1}} \cdots U_{x_2 x_1} U_{x_1 x_0} \psi_0(x_0) \ .$$

For the event of non-return, one must sum this expression over all trajectories x_j such that $x_j \neq 0$ for all $j > 0$. Evidently the resulting vector of components v_y is then given by a matrix product, of the form $(U^{-n}) V^n \psi_0$, where the matrix V is nothing but the matrix U with its first row set to zero. We can also set the first column of V to zero if we re-express v as $(U^{-n}) V^{n-1} \psi_1$, wherein ψ_1 is just $U\psi_0$ with its first entry set to zero. This way, V becomes effectively a 2×2 matrix.

Recall now that for three sites we have (with $\omega = 1^{1/3}$)

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega \\ \omega & 1 & \omega \\ \omega & \omega & 1 \end{pmatrix}$$

whence also

$$V = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \omega \\ 0 & \omega & 1 \end{pmatrix}$$

For these matrices, powers of U and V can both be evaluated in essentially the same manner, by writing U or V as a linear combination of orthogonal projectors. Taking U as exemplar, we obtain by adding and subtracting a multiple of the identity matrix to U:

$$U = \lambda(1 - P) + \sigma P \ ,$$

where

$$P = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad 1 - P = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \ ,$$

with $\lambda = (1-\omega)/\sqrt{3}$ and $\sigma = (1+2\omega)/\sqrt{3}$. In the same way, defining $Q = 1/2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (or more correctly as the 3×3 matrix with this as its lower right hand corner), we can obtain V in the form,

$$V = \lambda(1-Q) + \rho Q \ ,$$

where

$$Q = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad 1-Q = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \ ,$$

with $\rho = -\omega^2/\sqrt{3}$. It follows immediately that $U^n = \lambda^n(1-P) + \sigma^n P$ and $V^{n-1} = \lambda^{n-1}(1-Q) + \rho^{n-1}Q$. Noticing now that $|\rho| = 1/\sqrt{3} < 1$, while $|\lambda| = |\sigma| = 1$, we see that in the limit $n \to \infty$ we can drop the second term in V, without affecting $|R'\rangle$, i.e. without affecting whether the sequence of approximations $|R'_n\rangle$ converges or what it converges to. And noticing further that $P(1-Q) = 0$, we see that we can also drop that term in the product $U^{-n}V^{n-1}$, leaving the simple asymptotic form, $U^{-n}V^{n-1} \sim \lambda^{-n}(1-P)\lambda^{n-1}(1-Q) = (1/\lambda)(1-P)(1-Q)$. We thus obtain, modulo an exponentially small correction,

$$|R'_n\rangle = \frac{1}{\lambda}(1-P)(1-Q)\psi_1 \ .$$

This formula leads to a somewhat odd conclusion. With our original initial condition that the particle begins at 0, the components of ψ_1 are just the last two entries of the first column of U, namely $(\omega/\sqrt{3})(0,1,1)$, which evidently belongs to the kernel of $1-Q$. Hence the event of non-return is again precluded: $|R'\rangle = 0$; and once again $|R\rangle = |\Omega\rangle = (1,0,0)$. At first sight, this result might appear to confirm one's classical intuition, but in fact it seems to be a coincidence, at least if we take the 3-site hopper as typical. For almost any other choice of initial amplitudes than $(1,0,0)$, $|R'\rangle$ does not vanish! In particular, if the particle starts at site 2 instead of site 0, then the event that it fails to visit site 0 has the non-zero vector-measure $|R'\rangle = (1/3, -1/6, -1/6)$. (The quantal measure of this same event in the sense of [21] is then $\langle R'|R'\rangle$, or $1/6$.) The vector-measure of the complementary event that the particle does visit 0 is then $|R\rangle = |\Omega\rangle - |R'\rangle = (-1/3, 1/6, 7/6)$.

Our analysis of the 3-site case used tacitly the fact that the events R and R' are free of straddling cylinder-sets, for the same reason that stem-events are. Convenient though this is, it means that our example fails to illustrate condition (iii) in our definition of an evenly convergent sequence. It would be good to work out an example where (iii) does come into play, since doing so could indicate whether that condition is a reasonable one to have added, or whether on the contrary it tends to rule out events that one would want to include.

It would also be good to work out some physically interesting instances of our approximation procedure in the causal set case. One might begin, for example, with the event "originary" for the relatively simple dynamics of complex percolation.

8. Epilogue: does physics need actual infinity?

Does the description of nature require actual infinities? Or is a truly finitary physics possible, in which infinite sets would figure only as potentialities?

Inasmuch as the theories to which we have grown accustomed employ real numbers heavily, they thereby presuppose an actual infinity of cardinality $\aleph_1$, as emphasized in [22]. In itself, however, this seems more a matter of convenience than of principle, since one could imagine making do with rational numbers of a very fine but finite precision that could be made still finer as the need arose — in other words a potential infinity. ⋆

The other prominent continuum in present-day physics is of course spacetime. Non-relativistically, one could again imagine circumventing the actual infinities that continuous space and time seem to imply, but when it comes to relativistic field theories, the new requirement of *locality* appears to force strict continuity on us. Perhaps one could get by with only $\aleph_0$ points, say points with rational coordinates, but even that would still be an actual infinity.

Quantum gravity raises all these questions anew, of course. String theory and loop quantum gravity both presuppose background continua, at least in their current formulations. Causal dynamical triangulations and the "asymptotic safety" approach retain locality and presuppose the same type of continuum as classical gravity, albeit not as background.

With causal sets, the situation seems more fluid. On one hand, they transcend locality, but on the other hand they still maintain *covariance* in the sense of label-invariance, and that brings with it an "infrared" infinity, as discussed earlier. An important new feature, however, is that now the infinity is in some sense pure gauge: we need it only because we have introduced both an auxiliary time parameter and a space of "completed causets"

⋆ In writing '$\aleph_1$', I have adopted the continuum hypothesis, $\aleph_1 = 2^{\aleph_0}$, for ... notational reasons.

in order to give a precise meaning to the concept of sequential growth. Could it be that a manifestly covariant formulation of growth dynamics could dispense with this "last remaining infinity"? Limited to measure theoretic tools inherited from the classical theory of stochastic processes, we apparently lack the technical means to ask the question properly. As things stand, we can acknowledge at a minimum that being able to refer to completed causets is very convenient even if it ultimately turns out not to be physically necessary. (One can also comment here that the cardinality of a completed causet, though not finite, is reduced to that of the integers. On the other hand, the associated sample-space Ω still has the cardinality of the continuum.)

Based on this evidence, one could perhaps agree that physics is tending toward more finitary conceptions, even if it hasn't genuinely reached them yet. In particular, even if causal sets are implicitly free of actual infinities, the available mathematical tools don't let us express this fact clearly. Might some of the tools that we seem to lack arise naturally in the course of attempts, like those above, to extract well-defined generalized measures from quantal path-integrals and path-sums?

Appendix. Some symbols used, in approximate order of appearance

Ω (the sample-space or space of histories), $\Omega^{physical}$, Ω^{gauge}, $\Omega(n)$

$0 \subseteq \Omega$ (the empty subset)

$\mathrm{cyl}(c)$ (the cylinder event corresponding to the truncated history c)

$\mathfrak{Z}$ (the semiring of cylinder events), $\mathfrak{Z}_n$

$\mathfrak{S} = \mathfrak{R}\mathfrak{Z}$ (the Boolean algebra generated by $\mathfrak{Z}$ = the finite unions of cylinder sets)

$\mathfrak{S}_n$

$\mathfrak{T}$ (the tree of truncated histories), $\mathfrak{T}_n$

$\mathbf{1}^z \equiv \exp 2\pi i z$

$\bigvee \mathfrak{S}$, $\bigwedge \mathfrak{S}$, $\bigvee \bigwedge \mathfrak{S}$

$S(\alpha)$, $\widetilde{S}(\alpha)$

lim, Lim, lim inf, lim sup

$\prec$

$\mathbb{P}$, $\overline{\mathbb{P}}$

$A \sqcup B$

$AB = A \cap B$

$A + B = (A \cup B) \backslash (A \cap B)$

$|Z\rangle = \mu(Z)$

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