Boundary and Corner Terms in the Action for General Relativity

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Abstract

We revisit the action principle for general relativity, motivated by the path integral approach to quantum gravity. We consider a spacetime region whose boundary has piecewise C^2 components, each of which can be spacelike, timelike or null and consider metric variations in which only the pullback of the metric to the boundary is held fixed. Allowing all such metric variations we present a unified treatment of the spacelike, timelike and null boundary components using Cartan's tetrad formalism. Apart from its computational simplicity, this formalism gives us a simple way of identifying corner terms. We also discuss "creases" which occur when the boundary is the event horizon of a black hole. Our treatment is geometric and intrinsic and we present our results both in the computationally simpler tetrad formalism as well as the more familiar metric formalism. We recover known results from a simpler and more general point of view and find some new ones.

1 Introduction

The Einstein-Hilbert (EH) action for general relativity depends on the metric and its first and second derivatives. Indeed, the dependence on second derivatives is forced on us by the principle of general covariance since, there is no local coordinate scalar that can be formed from the metric and its first derivatives. By an appropriate choice of coordinates, we can make the first derivatives vanish at any point so that the only candidate for the action is the cosmological constant term.

While the EH Lagrangian does depend on the second derivatives of the metric, the dependence is rather innocuous since, as it turns out, the equations of motion are second order in metric derivatives, rather than fourth order, as one might naively expect. One can remove the dependence on second derivatives by adding a total divergence to the EH Lagrangian, which integrates to a boundary term. The appropriate action for general relativity is therefore the EH action with this boundary term. This makes the action first order in the metric derivatives: the second derivative term $\partial\partial g$ present in the Einstein Hilbert Lagrangian is replaced by a term of the form $(\partial g)^2$. All of this has been known for a while [3, 4].

Our first reason for revisiting the action principle for General Relativity is the path integral approach to quantum gravity. In summing over histories, we would like the quantum amplitudes to have the "folding" property:

$$K(X_1, X_3) = \int dX_2 K(X_1, X_2) K(X_2, X_3), \tag{1}$$

where X_1 and X_3 are initial and final states respectively and X_2 is an intermediate state which is summed over. In the metric representation X_1, X_3 represent the metrics on an initial and final spatial hypersurface $\Sigma_{1,3}$ and (Σ_2, X_2), an intermediate spatial geometry. We would clearly like the action to be additive under a decomposition of spacetime into pieces. There is a close relation between additivity of the action and having a first order Lagrangian. This can be clearly seen in a particle mechanics analogy. Consider the amplitude for a particle to go from x_0 at time t_0 to x_N at time $T = t_N$. $K(x_0, t_0; x_N, T)$. Introducing time slices at $t_k = k\epsilon = kT/N$, we have the skeletonised version of the path integral

$$K(x_0, t_0; x_N, T) = \int dx_1 dx_2 ... dx_{N-1} K(x_0, t_0; x_1, t_1) K(x_1, t_1; x_2 t_2) ... K(x_{N-1} t_{N-1}; x_N, T), \tag{2}$$

If the Lagrangian is first order, i.e. if L depends only on x and $\dot{x}$, the additivity of the action is immediate. One writes the short time propagator replacing $\dot{x}$ in the Lagrangian by $(x_{k+1} - x_k)/\epsilon$. This results in nearest neighbour couplings on the time lattice with the sites labeled by k. Decomposing the lattice into two parts separated by t_j gives us the folding property Eqn (1). However, for a second order Lagrangian $L(x, \dot{x}, \ddot{x})$, one needs *three* time steps in order to define $\ddot{x}$. E.g $\ddot{x}_k = (x_{k+1} + x_{k-1} - 2x_k)/\epsilon^2$. This brings in *next* nearest neighbour couplings on the time lattice, which spoils the additivity of the action.

A related point stems from the tensor nature of the gravitational field, which is not captured in the simple particle analogy above. In summing over histories that go from X_1 to X_3 via X_2 we allow all spacetime geometries, which on pullback agree with X_2. No further restriction needs to be placed on the metric. In particular, the components of the metric in directions transverse to the spacelike surfaces need not be held fixed. Textbook treatments (see [5, 6] for example) however hold *all* components of the metric fixed on the boundary, which is a stronger requirement. In a path integral, one typically sums over all paths without requiring continuity of all components of the metric across Σ_2. All we need is that the pullback of the four-metric to Σ_2 agrees with X_2.

Our second reason for revisiting the action principle is to explore boundaries of different signatures. A region in spacetime may have boundaries with components which are spacelike, timelike and null. There may also be corners where components of the boundary join. We present a formalism in which all these cases are derived in a transparent manner. The role of boundaries in gravitational physics has been increasing in recent years. Ideas relating bulk and boundary degrees of freedom have been discussed in the context of black hole entropy. One of the possible applications of our work is in black hole physics.

The need for adding a total divergence to the Einstein-Hilbert action was realised very early in the history of General Relativity[2]. The required boundary counterterm was given a geometric interpretation by York[3] and this line of thought was carried further by Gibbons and Hawking in their work on black hole thermodynamics. When the boundary has corners, there is a need for additional corner terms. These were first discussed by Sorkin and Hartle [7, 8], and subsequently by Hayward[9], Brill and Hayward[10] for timelike and spacelike boundaries. The need for a treatment of null boundaries was recognised by Parattu et al [11, 12]. There are also several contributions by Neiman[15, 16, 17, 18] and Epp[14]. Very recently Lehner et al [1] have given a detailed account of this problem. Our work differs from all these in several respects. We postpone a discussion of the differences to the concluding section.

Our treatment uses both the tetrad formulation and the metric formulation. We present a unified approach to the different boundary signatures. Indeed, as will appear below, this simplifies the calculation considerably. In section 2 we review some of the mathematical preliminaries. In Section 3 we present the tetrad formulation, which brings out the need for the corner terms and their explicit forms. In Section 4, we perform the whole calculation in the metric formulation, which is more familiar to readers. Section 5 contains a discussion and some open questions.

2 Mathematical Preliminaries

The spacetime manifold $(\mathcal{M}, g_{ab})$ is described by a Lorentzian metric g_{ab}, where a, b are spacetime indices going over $(0,1,2,3)$. Our signature is $(-+++)$. We begin with the Einstein-Hilbert action

$$I_{\rm EH} = \frac{1}{2}\int d^4x\sqrt{-g}R \tag{3}$$

for a spacetime $(\mathcal{M}, g_{ab})$, where $\partial\mathcal{M} = \cup_i \Sigma_i$ can have several piecewise C^2 components Σ_i whose normals n_{ia} are everywhere either timelike, spacelike or null. We have chosen units in which $8\pi G$ has been set to 1.

Consider a single component of the boundary $\Sigma \subset \partial\mathcal{M}$. When Σ is non-null, the unit normal n_a satisfies $n^a n_a = \epsilon$ where $\epsilon \equiv \pm 1$ depending on whether Σ is timelike or spacelike. When Σ is null the normal n_a is not unique, but for each n_a there is an equivalence class of null vectors l^a which satisfy $n_a l^a = -1$. In order to unify the treatment of the null and non-null cases, in addition to the normal n_a to Σ we will find it useful to define a *transverse* vector Q^a to Σ which does not lie in $\mathrm{T_p}\,\Sigma$. For non-null Σ, Q^a is proportional to n^a, i.e., the transverse and normal directions coincide upto a sign. For null Σ the natural choice is $Q^a \propto l^a$. It is this identification of the transverse vector Q^a which helps unify our treatment, rather than the normal vector n^a. For a smooth boundary in spacetime, for example, it is not the normal that gives a continuous or consistent definition of the outward direction, but rather the transverse vector, as shown in Figure 1. The metric can be decomposed into components along Σ and transverse to it, so that

$$\begin{aligned} g_{ab} &= h_{ab} + \epsilon n_a n_b \quad (\mathrm{Non-Null}) \\ g_{ab} &= \sigma_{ab} - l_a n_b - n_a l_b \quad (\mathrm{Null}) \end{aligned} \tag{4}$$

where h_{ab} is the induced metric on Σ and σ_{ab} is the induced metric on the spatial slice $\sigma \subset \Sigma$ normal to n_a.

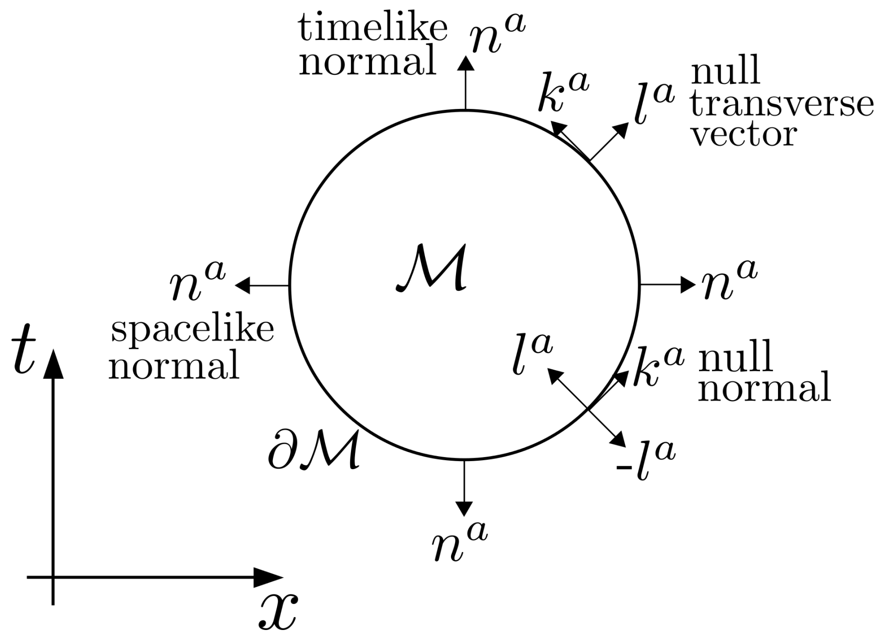


Figure 1: An illustration of how the normal and transverse vectors would be orientated on a patch of $1+1$ Minkowski spacetime whose boundary is a circle.

The "joins" or intersections $\mathcal{J}_{ij} = \Sigma_i \cap \Sigma_j$ of $\partial\mathcal{M}$ are allowed to be discontinuous in the sense that n_i^a and n_j^a differ at $\mathcal{J}_{ij}$. The $\mathcal{J}_{ij}$ are of codimension two and, like the boundary components, may also be timelike, spacelike or null. By considering the span of the normals n_i and n_j and looking at the range of the polynomial $f(\alpha) = (n_i + \alpha n_j)^2$ as α varies over the real line, one easily arrives at the following classification: the plane of the two normals in the tangent space has Lorentzian signature and the join is spacelike if i) at least one of the normals is timelike, or ii) both the normals are null, or iii) one normal is spacelike and the other null, with $n_i.n_j \neq 0$ or iv) both normals are spacelike and $n_i.n_j > 1$. The plane of normals has Riemannian signature and the join is timelike if i) both normals are spacelike and $n_i.n_j < 1$. Finally, the plane of normals is null and the join is null if i) one normal is null and the other space like with $n_i.n_j = 0$.

In Section 3 we use the Cartan tetrad formalism. This has the significant advantage offered by differential forms which can be integrated over manifolds without reference to a metric or its signature. It also has the advantage of giving us a fiducial Minkowski vector space as a reference. Given a metric g_{ab} on $\mathcal{M}$ we choose an orthonormal frame such that $g_{ab} = e_a^\mu e_b^\nu \eta_{\mu\nu}$. The tetrad e_a^μ maps a vector $X \in T_p\mathcal{M}$ to a point in $(\mathcal{M}_0, \eta_{\mu\nu})$

$$e_a^\mu : X \to e^\mu(X) = e_a^\mu X^a = X^\mu \in \mathcal{M}_0, \tag{5}$$

where $(\mathcal{M}_0, \eta_{\mu\nu})$ is a fixed fiducial Minkowski vector space, with a the spacetime index and μ the frame index ranging over $0, 1, 2, 3$. The map Eqn (5) is invertible, since we assume that the metric is non-degenerate. The spacetime metric g_{ab} is then the pullback of the fiducial metric $\eta_{\mu\nu}$ on $\mathcal{M}_0$. Frame indices μ, ν are raised and lowered with $\eta_{\mu\nu}$. There is an $O(1,3)$ gauge freedom in the choice of the e_a^μ. Associated with the e_a^μ are the connection 1-forms $A_{a\nu}^\mu = e_b^\mu \nabla_a e_\nu^b$ where ∇_a is the metric compatible Christoffel connection. A takes values in the Lie Algebra of $O(1,3)$ and is antisymmetric in the frame indices: $A^{\mu\nu} = -A^{\nu\mu}$. A is compatible with frames and satisfies Cartan's equation

$$de^{\mu} + A^{\mu}{}_{\nu} \wedge e^{\nu} = 0. \tag{6}$$

Written explicitly in the spacetime indices, the field strength of A is

$$F^{\mu\nu}_{ab} = \nabla_a A^{\mu\nu}_b - \nabla_b A^{\mu\nu}_a + A^{\mu}_{a\rho} A^{\rho\nu}_b - A^{\mu}_{b\rho} A^{\rho\nu}_a = 2\mathcal{D}_{[a} A^{\mu\nu}_{b]} \tag{7}$$

which is more succinctly expressed as $F^{\mu\nu} = dA^{\mu\nu} + A^{\mu}_{\rho} \wedge A^{\rho\nu} = 2\mathcal{D}A^{\mu\nu}$ where the wedge product is with respect to the spacetime indices.

Using the algebraic identity

$$\tilde{\eta}^{abcd} \epsilon_{\mu\nu\rho\lambda} e^{\rho}_a e^{\lambda}_b = -e(2!)^2 \, e^{[c}_{\mu} e^{d]}_{\nu} \tag{8}$$

where $e = \sqrt{-g}$ and $\tilde{\eta}^{abcd}$ is the Levi-Civita tensor density, the Einstein-Hilbert action[1] takes the form

$$I_{EH} = \frac{1}{4} \int d^4x \, \epsilon_{\mu\nu\rho\lambda} \;\; e^{\mu} \wedge e^{\nu} \wedge F^{\rho\lambda}. \tag{9}$$

The variation of I_{EH} gives us a bulk term (which yields the equations of motion) and a boundary term. The boundary term will be expressed as the variation of a boundary action $-I_B$, which gives us a counterterm to be added to the action. The total gravitational action is therefore

$$I_G = I_{EH} + I_B \tag{10}$$

where I_B in the non-null case is the Gibbons-Hawking-York (GHY) term. From its definition, the boundary term I_B is only defined up to terms that have zero variation. Certain imaginary terms that have been discussed before in the literature are of this variety. We will ignore them for the most part and comment on them in the conclusion. When the boundary is only piecewise C^2 the boundary contribution includes "corner" terms. It is the evaluation of these various boundary components in the tetrad formulation that we will now focus on.

3 The Tetrad formalism

3.1 Boundary Terms

Varying the Action I_{EH} we find

$$\delta I_{EH} = \frac{1}{4}\Big(2\int_{\mathcal{M}} \epsilon_{\mu\nu\rho\lambda} \, \delta e^{\mu} \wedge e^{\nu} \wedge F^{\rho\lambda} + 2\int_{\mathcal{M}} \epsilon_{\mu\nu\rho\lambda} \, e^{\mu} \wedge e^{\nu} \wedge D\delta A^{\rho\lambda}\Big). \tag{11}$$

The first term gives us Einstein's vacuum equations in the form $e^{\nu} \wedge F^{\rho\lambda}\epsilon_{\mu\nu\rho\lambda} = 0$ and the second term reduces to a boundary contribution

$$-\delta I_B = \frac{1}{2}\int_{\mathcal{M}} D(\epsilon_{\mu\nu\rho\lambda} \, e^{\mu} \wedge e^{\nu} \wedge \delta A^{\rho\lambda}) = +\frac{1}{2}\int_{\partial\mathcal{M}} \epsilon_{\mu\nu\rho\lambda} \, e^{\mu} \wedge e^{\nu} \wedge \delta A^{\rho\lambda} \tag{12}$$

[1]Note that we do not regard this as a first order Palatini action, since A is a function of e^{μ}_a determined by Eqn (6) and is not independent.

In order to calculate the boundary term we require that the pullback of the metric to the boundary $\mathcal{M}$ is unvaried. We can, in addition, demand that the pullback of e^μ to the boundary $\mathcal{M}$ also has zero variation. This permits us to take δ outside the integral and express it as the variation of

$$-I_B = \frac{1}{2}\int_{\partial\mathcal{M}} \epsilon_{\mu\nu\rho\lambda}\, e^\mu \wedge e^\nu \wedge A^{\rho\lambda}. \tag{13}$$

Our derivation so far is independent of the type of boundary $\partial\mathcal{M}$. We will now show that this expression is the GHY term written in a universal form, by looking at the three types of boundaries – spacelike, timelike and null. We now choose *adapted tetrads* so that one of the 1-form fields e^μ is normal to the boundary. The natural choices for the spacelike, timelike and null normals to $\partial\mathcal{M}_{s,t,n}$ are $e^0_a = n_a$, $e^1_a = n_a$ and $e^\pm_a = n_a$, respectively, where $e^\pm_a = (e^0_a \pm e^1_a)/\sqrt{2}$. In general we write $n_a = e^\alpha_a$ where $\alpha = 0, 1, \pm$ depending on $\partial\mathcal{M}$. Let us also always choose n^a to be outward directed for $\partial\mathcal{M}$, a well-defined concept for the non-null normals. For null normals, it is the transverse null tetrad l_a defined by $l_a n^a = -1$ which must be outward pointing. This picks the orientation of n_a. For example in Minkowski spacetime if $n_a = e^+_a$, then the transverse null tetrad $l_a = e^-_a$ is is chosen with the same time orientation as e^+_a so that $e^+_a e^{a-} = -1$.

Using the relation Eqn (8) we see that the integrand in Eqn (13) can be simplified. Using the notation $n_a = e^{\hat\alpha}_a$, where the hat indicates that α is a fixed index $(0, 1, \pm)$, for which the summation convention does not apply, we have

$$e n_a e^a_\mu e^b_\nu A^{\mu\nu}_b = e\delta^{\hat\alpha}_\mu e^b_\nu A^{\mu\nu}_b = e e^b_\nu A^{\hat\alpha\nu}_b \tag{14}$$

where the sum over ν extends over all indices *except* $\hat\alpha$ because of the antisymmetry of A in the frame indices. Putting in the form of A we have

$$-e e^b_\nu e^\nu_c \nabla_b e^{\hat\alpha c} = -e(\delta^b_c - e^b_{\hat\alpha} e^{\hat\alpha}_c)\nabla_b n^c, \tag{15}$$

which gives the universal boundary term

$$I_B = \int_{\partial\mathcal{M}} e(g^{bc} - e^b_{\hat\alpha} e^{\hat\alpha c})\nabla_b n_c. \tag{16}$$

Note that we have made no assumption above regarding extending the normal n_a off the boundary. The normal is only defined at points on the boundary and we only use its tangential derivatives.

Observe further that in the adapted tetrads for non null normals ($\hat\alpha = 0, 1$)

$$e^b_{\hat\alpha} e^{\hat\alpha c} = \eta_{\hat\alpha\hat\alpha} e^{\hat\alpha b} e^{\hat\alpha c} = \epsilon n^b n^c \tag{17}$$

and for null normals ($\hat\alpha = +$)

$$e^b_{\hat\alpha} e^{\hat\alpha c} = \eta_{+-} e^{-b} e^{+c} = -l^b n^c \tag{18}$$

Using the decomposition Eqn (4) we note that

$$(g^{bc} - e^b_{\hat\alpha} e^{\hat\alpha c})\nabla_b n_c = (g^{bc} - \epsilon n^b n^c)\nabla_b n_c = h^{bc}\nabla_b n_c = K \quad (\mathrm{Non-Null}) \tag{19}$$

$$(g^{bc} - e^b_{\hat\alpha} e^{\hat\alpha c})\nabla_b n_c = (\sigma^{bc} - n^b l^c)\nabla_b n_c = (\Theta - \kappa) \quad (\mathrm{Null}) \tag{20}$$

(21)

where $K = h^{ab}\nabla_a n_b$ is the extrinsic curvature of $\partial\mathcal{M}_{t,s}$, $\Theta = \sigma^{ab}\nabla_a n_b$ the null expansion on $\partial\mathcal{M}_n$ and the surface gravity $\kappa = l^a n^b \nabla_b n_a$ measures the failure of n^a to be affinely parameterised. For $\partial\mathcal{M}_{s,t}$ this gives the expected GHY term

$$I_B = \int \sqrt{\pm h} K \, d^3x, \tag{22}$$

where x's are coordinates on the boundary. For $\partial\mathcal{M}_n$ this gives

$$I_B = \int \sqrt{\sigma}(\Theta - \kappa) d\lambda d^2x, \tag{23}$$

where the x's are now spatial coordinates on the null surface and λ is a parameter along the null generator satisfying $n^a\partial_a = \frac{\partial}{\partial\lambda}$.

The boundary term Eqn (13) is not gauge invariant under $O(1,3)$ transformations (although its variation is). This is because A transforms inhomogenously by

$$A \to \Lambda^{-1} A \Lambda + \Lambda^{-1} d\Lambda \tag{24}$$

with the result that

$$I_B \to I_B + \frac{1}{2}\int_{\partial\mathcal{M}} \epsilon_{\mu\nu\rho\lambda}\, e^\mu \wedge e^\nu \wedge \mathbf{g}^{\rho\lambda} \tag{25}$$

where $\mathbf{g} = \Lambda^{-1} d\Lambda$ is in the Lie Algebra of $O(1,3)$.

We note that in the adapted tetrads there is a residual gauge freedom in the little group H, which preserves the normal. The little group is given by $H = O(3)$ for timelike, $H = O(1,2)$ for spacelike and $H = E(2)$ for null normals. It is easily checked that the adapted boundary term *is* invariant under gauge transformations of the little group. In fact for $\Lambda \in H$, $\mathbf{h} = \Lambda^{-1} d\Lambda$ satisfies $\mathbf{h}^{\hat{\alpha}\lambda} = \mathbf{h}^{\rho\hat{\alpha}} = 0$ for $\hat{\alpha}$ a fixed index labelling the normal, as above. For vector fields t^a, tangent to the boundary, we have $e^{\hat{\alpha}}(t) = n_a t^a = 0$ and so the change in I_B under a gauge transformation

$$\Delta I_B = \frac{1}{2}\int_{\partial\mathcal{M}} \epsilon_{\mu\nu\rho\lambda}\, e^\mu \wedge e^\nu \wedge \mathbf{h}^{\rho\lambda} \tag{26}$$

vanishes entirely, since the four indices of $\epsilon_{\mu\nu\rho\lambda}$ must all be distinct for a nonvanishing contribution.

We introduce four discrete elements D of $O(1,3)$ corresponding to each of the connected components of the group. They are I, P, T and PT, where P and T stand for parity and time-reversal respectively. Since these are constant matrices, the connection A transforms homogeneously and the boundary term Eqn (13) is invariant under such transformations. These discrete elements D will be needed in the next section to relate frames across a join.

3.2 Corner Terms

The fact that the boundary term Eqn (13) is not gauge invariant can be exploited to identify the corner terms. By adapting our frame to the normal we have been able to derive the forms Eqn (22,23) of the boundary GHY terms for all signatures of the boundary. When there is a join of two boundary components, the adapted frames will not, in general, agree at the join. In order to pass from one frame to the other we will use the following procedure. By means of a gauge transformation in the little group H, we will ensure that two of the frame fields from each boundary component are tangent to the join and agree with each other at the join. By use of discrete elements in $O(1,3)$, we will ensure that the frames are related by an element in the identity component of $O(1,3)$. With these choices, the relation between the two frames is a Lorentz transformation in the 2-dimensional plane of normals. The change in the boundary term Eqn (13) under this $O(1,3)$ gauge transformation gives us the corner terms.

Let Σ_i and Σ_j meet along a join $\mathcal{J}_{ij}$. The boundary term Eqn (16) is valid when an adapted frame is used, but the latter changes in going from Σ_i to Σ_j. This corresponds to effecting an $O(1,3)$ transformation in Eqn (13) which relates the adapted frames $e^\mu_{\rm (i)}, e^\mu_{\rm (j)}$ of Σ_i, Σ_j. By operating on the frames by discrete elements D of $O(1,3)$, we can arrange that the two frames are related by an element in the identity component of $O(1,3)$.

For spacelike joins, by further gauge transformations in the little group, one can arrange that $e^2_{\rm (i)} = e^2_{\rm (j)}$ and $e^3_{\rm (i)} = e^3_{\rm (j)}$ and that both of these are orthogonal to the timelike plane of normals. The two frames $e_{\rm (i)}$ and $e_{\rm (j)}$ are therefore related by a Lorentz boost in the timelike plane of normals,

$$e^\mu_{\rm (i)} = \Lambda_{\rm (ij)}{}^\mu{}_\nu e^\nu_{\rm (j)}. \tag{27}$$

We define the discontinuous gauge transformation $\lambda \in O(1,1)$ to be the identity on Σ_i and $\Lambda_{\rm (ij)}$ on Σ_j

$$\lambda_{ij} = \exp\,[\eta K \Theta^{(H)}_{ij}], \tag{28}$$

where $\Theta^{(H)}_{ij}$ is the Heaviside function that takes values 0 on Σ_i and 1 on Σ_j, η is the rapidity parameter and K the boost generator in the plane of normals. $\mathbf{g}^{\rho\sigma} = (\Lambda_{\rm (ij)}{}^{-1} d\Lambda_{\rm (ij)})^{\rho\sigma} = \eta K^{\rho\sigma} d\Theta^{(H)}_{ij}$ is therefore proportional to a delta function that is peaked on the join $\mathcal{J}_{ij}$ and vanishes on Σ_i and Σ_j. The gauge transformation of the boundary term Eqn (25) results in the join term

$$\frac{1}{2}\int_{\mathcal{J}_{ij}} \eta \epsilon_{\mu\nu\rho\sigma} e^\mu \wedge e^\nu K^{\rho\sigma}, \tag{29}$$

which in this case (since only $K^{01} = -K^{10}$ is non vanishing), simplifies to

$$I_{\mathcal{J}_{ij}} = \int_{\mathcal{J}_{ij}} e^2 \wedge e^3 \eta = \int_{\mathcal{J}_{ij}} dA\eta, \tag{30}$$

where dA is the area element of the join.

It is possible to express the rapidity that appears in the corner term using the angle between the normals. The Lorentz boost with rapidity η can be written as $e^+_{\rm (j)} = (\exp\eta) e^+_{\rm (i)}, e^-_{\rm (j)} = (\exp -\eta) e^-_{\rm (i)}$ and the timelike and spacelike normals as $n_{\rm (i,j)} = (e^+_{\rm (i,j)} \pm e^-_{\rm (i,j)})/\sqrt{2}$, respectively. Using the symbols T, S, N to denote a timelike , spacelike or null normal, respectively, we find that if the two normals at the join are (i) TT: $n_i.n_j = -\cosh\eta$, (ii) TS: $n_i.n_j = \sinh\eta$, (iii) TN: $n_i.n_j = -\exp\eta/\sqrt{2}$, (iv) SS: $n_i.n_j = \cosh\eta$, (v) SN: $n_i.n_j = \exp\eta/\sqrt{2}$ and (vi) NN:$n_i.n_j = -\exp\eta$.

For timelike joins, the argument is very similar. We can by gauge transformations in the little group arrange that $e^0_{\rm (i)} = e^0_{\rm (j)}$ and $e^1_{\rm (i)} = e^1_{\rm (j)}$ and that both of these are orthogonal to the spacelike plane of normals. The two frames $e^\mu_{(\mathrm{i})}$ and $e^\mu_{(\mathrm{j})}$ are now related by a rotation in the spacelike plane of normals

$$e^\mu_{(\mathrm{i})} = \Lambda_{(\mathrm{ij})}{}^\mu{}_\nu e^\nu_{(\mathrm{j})}. \tag{31}$$

Again, define the discontinuous gauge transformation $\lambda \in O(2)$ as the identity on Σ_i and $\Lambda_{(\mathrm{ij})}$ on Σ_j, so that

$$\lambda_{ij} = \exp\,[\eta J \Theta^{(H)}_{ij}], \tag{32}$$

where η is now the rotation angle and J the rotation generator in the plane of normals. Again this gives rise to a contribution from the join

$$\frac{1}{2}\int_{\mathcal{J}_{ij}} \eta \epsilon_{\mu\nu\rho\sigma} e^\mu \wedge e^\nu J^{\rho\sigma}. \tag{33}$$

Since the nonvanishing components of J are $J^{23} = -J^{32}$, we have the form of the corner term:

$$I_{\mathcal{J}_{ij}} = \int_{\mathcal{J}_{ij}} e^0 \wedge e^1 \eta = \int_{\mathcal{J}_{ij}} dA\eta, \tag{34}$$

where dA is the area element of the join. Relating the inner products to the angles follows the case of spacelike joins and we do not repeat the analysis here. A salient difference is that the angles are only defined modulo 2π. This arises because the group $SO(1,1)$ is simply connected ($\pi_1(SO(1,1)) = 0$), while the group $SO(2)$ is multiply connected ($\pi_1(SO(2)) = \mathbb{Z}$). This ambiguity does not however affect the variation.

Null joins differ in that the plane of normals and the tangent space to the join share a one dimensional, null subspace. If n_i is spacelike and n_j is null (with $n_i.n_j = 0$), n_j belongs *both* to the span of normals and the tangent space to the join. It is possible to adapt a null Lorentz frame to both Σ_i and Σ_j as follows: $e^+_i = e^+_j = n_j$, $e^3_i = e^3_j = n_i$ and $e^2_i = e^2_j$, $e^-_i = e^-_j$. Since $e^\mu_i = e^\mu_j$, we have $\Lambda_{(\mathrm{ij})}$ equal to the identity and $\eta = 0$. The corner term therefore vanishes.

3.3 Creases

A physically interesting situation covered by the above analysis occurs when one of the boundaries of spacetime is the event horizon of a dynamically evolving black hole. In this case the horizon does not remain smooth when new generators enter or leave the horizon. Suppose that we are interested in the boundary of a future set. (The case of past sets is similar). The boundary of a future set is ruled by null generators. However, when these null generators cross because of gravitational focussing effects, they leave the boundary and enter into the interior of the future set. The horizon then develops a caustic, generically a spacetime region of codimension two, where the normal to the wavefront is discontinuous. When this happens, we have a "crease" which separates regions of the null surface with different normal vectors. Locally, this is no different from a null-null join discussed above. From the analysis already presented we would expect a boundary term to appear as an integral along the crease of the rapidity parameter, just as in the NN case treated above.

4 The metric formalism

While the tetrad formulation is calculationally simpler, it is also true that the metric formulation is more familiar to most readers. In this section, we present the metric formulation of the above calculation, which has also recently been given in [1]. To find the boundary contribution to the action we need to consider the most general class of variations δg^{ab} which leave the induced metric on Σ fixed, so that for any $t^a, s^a \in \mathrm{T}\,\Sigma$,

$$\delta g_{ab} t^a s^b = 0. \tag{35}$$

Using $\delta(g_{ab} g^{ac}) = 0$ to relate the variation of the covariant and contravariant metrics we find that

$$\delta g_{ab} = -g_{bd} g_{ac} \delta g^{cd} \;\Rightarrow\; \delta g^{ab} t_a s_b = 0. \tag{36}$$

From the decomposition Eqn (4) of g_{ab} into components transverse to and along Σ, we see that the most general variation takes the form

$$\delta g^{ab} = 2 n^{(a} \delta Q^{b)}, \tag{37}$$

where we have made the identification

$$Q^a = \begin{cases} \epsilon\, n^a & (\mathrm{Non-null}) \\ -\, l^a & (\mathrm{Null}) \end{cases} \tag{38}$$

The 4-vector δQ^a therefore gives the full admissible $10 - 6 = 4$ parameter degrees of freedom in this class of variations.

It is useful to decompose δQ^a into components transverse to and along Σ

$$\delta Q^a = \alpha Q^a + t^a, \tag{39}$$

where $t^a \in \mathrm{T}\,\Sigma$, and such that $n^a t_a = 0$ for both null and non-null cases. When Σ is null t^a can be further decomposed as

$$t^a = \beta n^a + s^a \quad (\mathrm{Null}) \tag{40}$$

where $s^a n_a = 0$.

Using the unperturbed metric to raise and lower indices, the variation of the covariant quantities is

$$\delta Q_a = \delta g_{ab} Q^b + g_{ab} \delta Q^b, \quad \delta n_a = \delta g_{ab} n^b + g_{ab} \delta n^b, \tag{41}$$

which along with Eqns (36-40) simplifies to the general expression

$$\delta n_a = -\alpha n_a, \tag{42}$$

$$\delta Q_a = \beta n_a, \tag{43}$$

where α, β are independent when Σ is null and $\beta = -\epsilon\alpha$ when Σ is non-null. The parameter α can moreover be related to a variation of the volume element in both the null and non-null cases

$$\alpha = -\delta(\ln \sqrt{-g}) \tag{44}$$

where we have used $\delta(\ln\sqrt{-g}) = -\frac{1}{2}g_{ab}\delta g^{ab} = -\alpha g_{ab}n^aQ^b$ using Equations (37) and (39). The boundary term resulting from the variation of the Einstein-Hilbert action has the general form

$$-\delta I_B = \frac{1}{2}\int_\Sigma dV v^a n_a \tag{45}$$

where $v^a = -g^{ab}C^c_{cb} + g^{cb}C^a_{cb}$ and dV is the volume element on Σ and C^c_{ab} is the variation in the metric compatible connection

$$C^c_{ab} = \frac{1}{2}g^{cd}\{\nabla_a\delta g_{bd} + \nabla_b\delta g_{ad} - \nabla_d\delta g_{ab}\}. \tag{46}$$

with ∇_a the connection compatible with g_{ab}.

The task is then to find the boundary term I_B which has to be added to the Einstein-Hilbert action. The integrand in Eqn (45) can be simplified to

$$v^a n_a = -n^a g^{bc}\{\nabla_a\delta g_{bc} - \nabla_b\delta g_{ac}\}, \tag{47}$$

for all types of Σ. We now examine the two separate cases.

4.1 Σ non-null

Using $g^{ab} = h^{ab} + \epsilon n^a n^b$ reduces Eqn (47) to

$$v^a n_a = -n^a h^{bc}\{\nabla_a\delta g_{bc} - \nabla_b\delta g_{ac}\}. \tag{48}$$

Comparing with the variation of the extrinsic curvature K of Σ we see that

$$\begin{aligned} -2\delta K &= 2h^{ab}C^c_{ab}n_c - 2h^{ab}\nabla_a\delta n_b \\ &= -n^a h^{bc}\nabla_a\delta g_{bc} + 2n^a h^{bc}\nabla_b\delta g_{ac} + 2\alpha K. \end{aligned} \tag{49}$$

The first terms in Eqn (48) and Eqn (49) are the same. In [5, 6] the second term in Eqn (48) and the remaining terms in Eqn (49) are put to zero but this unnecessarily restricts the allowed variations. Allowing the full 4-parameter variation the second term in Eqn (48) reduces to

$$n^a h^{bc}\nabla_b\delta g_{ac} = -2\alpha K - h^{ab}\nabla_a t_b, \tag{50}$$

so that

$$v^a n_a = -2\delta K + h^{ab}\nabla_a t_b. \tag{51}$$

Thus, in agreement with the standard results in [5, 6]

$$-\delta I_B + \delta I_K = \frac{1}{2}\int_\Sigma d^3x\sqrt{\epsilon h}D_a t^a, \tag{52}$$

where D_a is the connection compatible with h_{ab} and

$$I_K = \int_\Sigma \sqrt{\epsilon h}K. \tag{53}$$

If $\mathcal{J}_i \subset \Sigma$ are either spacelike or timelike "corners" of Σ with normals $m^a_{(i)} \in T\Sigma$, the variation Eqn (52) reduces to

$$\frac{1}{2}\sum_i \int_{\mathcal{J}_i} d^2x\sqrt{\epsilon' q}\; t^a m_{(i)a} \equiv \sum_i \delta I_{\mathcal{J}_i} \tag{54}$$

where q_{ab} is the induced metric on $\mathcal{J}_i$ and $\epsilon' = \pm 1$ depending on whether $\mathcal{J}_i$ is spacelike or timelike. If $\mathcal{J}_i$ is null, then

$$\delta I_{\mathcal{J}_i} = \frac{1}{2}\int_{\mathcal{J}} dx d\lambda \sqrt{\tilde{q}}\; t^a j_a \tag{55}$$

where $\sqrt{\tilde{q}}$ is the volume element on the 1 dimensional spatial section of $\mathcal{J}$ and j^a its null normal. As we will see in the next few sections, such corner terms will not contribute. Thus, the boundary term to be added to the action is

$$I_B = I_K - \sum_i I_{\mathcal{J}_i} \tag{56}$$

where $I_{\mathcal{J}_i}$ are the yet to be determined corner terms.

4.2 Σ null

Since the null geodesics generated by n^a are hypersurface orthogonal, we can suppose that they satisfy the condition $\nabla_{[a}n_{b]} = 0$. Combining this with the variation $\delta g_{ab} = 2n_{(a}g_{b)c}\delta l^c$ allows us to simplify Eqn (47) to

$$v^a n_a = n^a\nabla_a\alpha - \alpha\Theta + 2\alpha\kappa, \tag{57}$$

where Θ and κ are the null expansion and surface gravity of Σ respectively.

The natural analog of the GHY term is

$$I_\Theta = \int_\Sigma d^2x d\lambda \sqrt{\sigma}\Theta \tag{58}$$

and it is therefore natural to first compare this variation with Eqn (57). Since $n^a = (\partial/\partial\lambda)^a$ remains invariant under this class of variations the affine parameter λ is unchanged, so that δI_Θ again only involves the integrand Θ. While $\delta\sigma_{ab} = 0$,

$$\delta\sigma^{ab} = \delta g^{ac}g^{bd}\sigma_{cd} + g^{ac}\delta g^{bd}\sigma_{cd} = 2n^{(a}s^{b)} \tag{59}$$

where we have used Equations (39) (40) so that

$$\begin{aligned} 2\delta\Theta &= 4n^{(a}s^{b)}\nabla_a n_b - 2\sigma^{ab}C^c_{ab}n_c + 2\sigma^{ab}\nabla_a\delta n_b \\ &= -4\kappa s^b n_b - 2\alpha\Theta + 2\alpha\Theta = 0. \end{aligned} \tag{60}$$

Given the form of Eqn (57) it is therefore clear that an additional boundary piece is required. Instead, consider (see [11, 12])

$$I_\kappa = \int_\Sigma d^2x d\lambda \sqrt{\sigma}\kappa, \tag{61}$$

whose variation again only involves the integrand κ,

$$2\delta\kappa = 2(\delta l^a n^b\nabla_b n_a - l^a n^b C^c_{ab}n_c + l^a n^b\nabla_b\delta n_a)$$

$$= \quad 2\alpha\kappa + 2n^b\nabla_b\alpha. \tag{62}$$

Thus

$$\begin{aligned} -\delta I_B &= \delta I_\kappa - \frac{1}{2}\int_\Sigma d^2x d\lambda \sqrt{\sigma}(n^a\nabla_a\alpha + \alpha\Theta) \\ &= \delta I_\kappa - \frac{1}{2}\int_\Sigma d^2x d\lambda \frac{d(\sqrt{\sigma}\alpha)}{d\lambda} \\ &= \delta I_\kappa + \delta I_{\mathcal{J}}(\lambda_i) - \delta I_{\mathcal{J}}(\lambda_f) \end{aligned} \tag{63}$$

where we have defined

$$\delta I_{\mathcal{J}}(\lambda) \equiv \frac{1}{2}\int_{\mathcal{J}} d^2x \sqrt{\sigma(\lambda)}\alpha(\lambda) \tag{64}$$

and have used the expression $\Theta = \frac{1}{\sqrt{\sigma}}\frac{d\sqrt{\sigma}}{d\lambda}$. Here, $\lambda_{i,f}$ are the initial and final values of λ at the spacelike boundaries, $\mathcal{J}_i, \mathcal{J}_f$ of Σ. As one can see, it is *only* such spacelike corner terms that contribute for Σ null; there is no contribution from a null corner. The boundary term to be added to the action is therefore

$$I_B = -I_\kappa + I_{\mathcal{J}}(\lambda_f) - I_{\mathcal{J}}(\lambda_i), \tag{65}$$

where $I_{\mathcal{J}}(\lambda)$ is a yet to be determined null corner term contribution. At this point I_Θ can also be included, though its variation vanishes. This brings the boundary term into the same form as that obtained in the tetrad formulation.

Before moving on to a calculation of the corner terms it is worthwhile saying a little about the question of uniqueness of the transverse vector Q^a. In the non-null case, it is easy to find a unique transverse vector. For any timelike or spacelike vector $r^a \in T_pM$ we associate a unique transverse subspace $R \subset T_p\mathcal{M}$ such that $r.v = 0, \forall v^a \in R$. r^a is then transverse to Σ and if it is normalised to ± 1 it is the unique unit normal n^a. In the null case, the situation is a little more complicated since $l^a n_a = -1$ does not give a unique l^a associated to every n^a. We can however enforce uniqueness as follows. If m_1^a, m_2^a are spacelike unit vectors in $T_p\Sigma$ such that $n.m_{1,2} = 0$, let M_1, M_2 be their associated transverse subspaces, respectively. Then $l^a \in M_1 \cap N_1$ is the unique transverse null vector satisfying $l^a n_a = -1$.

4.3 Corner Terms for Null-Null Boundary

The intersection $\mathcal{J}$ of two null hypersurfaces $\Sigma_{1,2}$ can be either spacelike or null. Examples of these are shown in Figure 2. The achronality of a null hypersurface precludes the intersection from being timelike. When $\mathcal{J}$ is null, as we have seen there is no corner contribution for null Σ. Indeed, in any case, such a null intersection is not a join as per our definition in Section 2. We therefore need only consider the case when $\mathcal{J}$ is spacelike. For clarity in this section we will resort to using k^a to depict the null normal and leave n^a to denote the non-null normal.

Let the normals to the two null boundary components $\Sigma_{1,2}$ be $k^a_{1,2}$. In order to fix the relative signs of the corner terms Eqn (64) it is important to define first what is meant by an outward pointing null normal. We will define this using the transverse vector l^a rather than the normal k^a. For a join arising in the causal diamond shown in Figure 2 the join is *outward convex* in

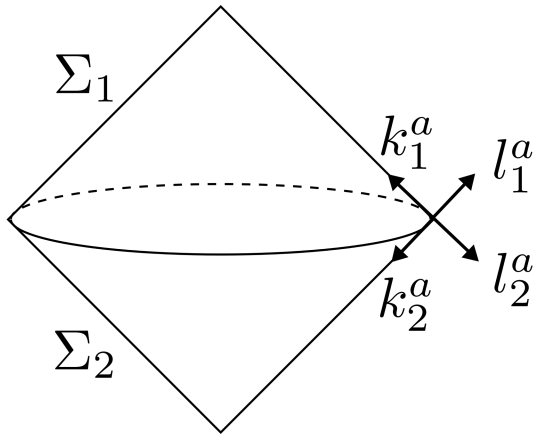


Figure 2: An interval in $2+1$ Minkowski spacetime.

the following sense. The outward pointing transverse vector l_1^a to Σ_1 in T$\mathcal{J}$ is along the positive $(\partial/\partial v)^a$ direction. Hence k_1^a is along the positive $(\partial/\partial u)^a$ direction. On the other hand, l_2^a for Σ_2 is in the negative $(\partial/\partial u)^a$ direction, which makes k_2^a lie in the negative $(\partial/\partial v)^a$ direction. Thus, in this case the parameters $\lambda_{1,2}$ on $\Sigma_{1,2}$ both take their initial values on $\mathcal{J}$. This is an outward convex join. Conversely at an *outward concave* join $\lambda_{1,2}$ both take their final values on $\mathcal{J}$.

Thus, from Eqns (64) and (65) the total contribution to $\mathcal{J}$ is

$$\delta I_{\mathcal{J}} = \pm\frac{1}{2}\int_{\mathcal{J}} d^2x\sqrt{\sigma}(\alpha_1+\alpha_2) \tag{66}$$

depending on whether the join is concave or convex. Here $\alpha_{1,2}$ come from the variations of l_1^a and l_2^a on $\mathcal{J}$. The variations of the metric on Σ_1 and Σ_2 are, respectively

$$\delta g_{1,2}^{ab} = k_{1,2}^{(a}(\alpha_{1,2}l_{1,2}^{b)} + \beta_{1,2}k_{1,2}^{b)} + s_{1,2}^{b)}) \tag{67}$$

which at $\mathcal{J}$ must match up, i.e.,

$$\delta g_1^{ab}|_{\mathcal{J}} = \delta g_2^{ab}|_{\mathcal{J}}. \tag{68}$$

Of the 4 null vectors $k_{1,2}^a, l_{1,2}^a$, we pick two linearly independent ones to be the normals $k_{1,2}^a$, so that $l_{1,2}^a = u_{1,2}k_1^a + v_{1,2}k_2^a$. Using $l_1.l_1 = l_2.l_2 = 0$, $l_{1,2}.k_{1,2} = -1$, $v_1 = u_2 = -\frac{1}{k_1.k_2}$ and $u_1 = v_2 = 0$, which means that $l_{1,2}^a = -\frac{1}{k_1.k_2}k_{2,1}^a$. Denoting the equality on $\mathcal{J}$ by $\doteq$, Eqn (68) then implies that

$$\begin{aligned} \alpha_1 &\doteq \alpha_2 \\ \beta_1 &\doteq \beta_2 \\ s_1^a \doteq s_2^a &\doteq 0 \end{aligned} \tag{69}$$

where we have used the linear independence of k_1^a and k_2^a and the directions tangent to $\mathcal{J}$. Noting that $\delta(k_1.k_2) = k_1.\delta k_2 = -\alpha_2(k_1.k_2)$ allows us to express α_1 as the variation $\alpha_1 = -\delta(\ln(|k_1.k_2|)$ so that the corner term can be written as

$$I_{\mathcal{J}} = \mp\int d^2x\sqrt{\sigma}\ln(|k_1.k_2|), \tag{70}$$

with the sign depending on whether $\mathcal{J}$ is concave or convex outward.

4.4 Corner Terms for Null-Spacelike or Null-Timelike Boundary

The null-spacelike join can only be spacelike, while the null-timelike join can be either spacelike or null. We will first consider the case when $\mathcal{J}$ is spacelike. If Σ_1 is null and Σ_2 non-null, the corner contributions to the spacelike join $\mathcal{J}$ come from Eqn (54) and (64) so that

$$\delta I_{\mathcal{J}} = \pm\frac{1}{2}\int_{\mathcal{J}} d^2x\sqrt{\sigma}\alpha_1 - \frac{1}{2}\int_{\mathcal{J}} d^2x\sqrt{\sigma}t^a m_a, \tag{71}$$

where m^a is normal to $\mathcal{J}$ in Σ_2 and the $\pm$ sign in front of the first term is positive or negative if it is an initial or final boundary, respectively, with respect to the outward directed normal to Σ_1. Again, we will see that $\mathcal{J}$ can be thought of as concave or convex outward, and this determines the sign of the first term, but also of the second term.

The variations of the metric on Σ_1 and Σ_2 are

$$\begin{aligned} \delta g_1^{ab} &= k^{(a}(\alpha_1 l^{b)} + \beta_1 k^{b)} + s_1^{b)}) \\ \delta g_2^{ab} &= n^{(a}(\alpha_2 n^{b)} + t^{b)}). \end{aligned} \tag{72}$$

Decomposing $t^a \in \mathrm{T}\,\Sigma_2$ $t^a = r^a + s_2^a$, where $s_2^a \in \mathrm{T}\mathcal{J}$ and r^a is transverse to $\mathrm{T}\mathcal{J}$, and using the k^a, l^a basis we express n^a, r^a (both transverse to $\mathrm{T}\mathcal{J}$) as $n^a = u_1 k^a + v_1 l^a$, $\quad r_2^a = u_2 k^a + v_2 l^a$, where the normalisation $n^a n_a = \epsilon \Rightarrow v_1 = \epsilon\frac{1}{2u_1}$. Using the matching condition Eqn (68) and the fact that v_1 can be arbitrary, we find that

$$\begin{aligned} \alpha_2 &\doteq -\frac{v_2}{v_1} \doteq -2\epsilon u_1 v_2 \\ \alpha_1 &\doteq v_1(\alpha_2 u_1 + u_2) \\ \beta_1 &\doteq u_1(\alpha_2 u_1 + u_2) \doteq 2\epsilon u_1^2 \alpha_1 \\ s_1^a \doteq s_2^a &\doteq 0. \end{aligned} \tag{73}$$

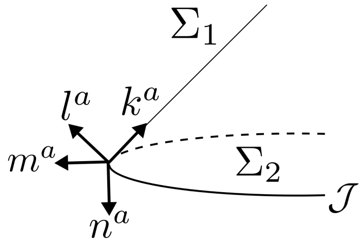
Since the normal to a spacelike $\mathcal{J}$ in $\mathrm{T}\,\Sigma_2$ is spacelike when Σ_2 is spacelike and timelike when Σ_2 is timelike, $m^a m_a = -\epsilon$. Combining this with $n^a m_a = 0$, we can express $m^a = \epsilon|u_1|k^a + \frac{1}{2|u_1|}l^a$. Here we use the fact that since m^a is outward directed with respect to Σ_2, it is ϵ times the sense of the outward directed k^a as shown in Figures 3 and 4.

Thus,

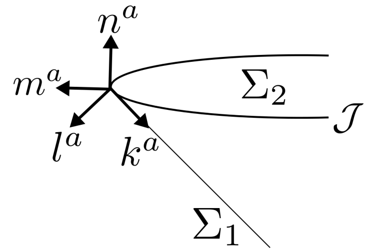
$$t^a m_a = \frac{\epsilon}{2|u_1|}(\alpha_2 u_1 + u_2) = \alpha_1\frac{|u_1|}{u_1}, \tag{74}$$

where $\frac{|u_1|}{u_1} = \pm 1$ depending on the orientation of Σ_2 with respect to Σ_1. Specifically, $u_1 = n.l$, the inner product of the transverse vectors (which determine the "outward" directions) of Σ_1 and Σ_2. When $u_1 < 0$, $\mathcal{J}$ is an initial boundary with respect to the affine parameter λ on Σ_1, and when $u_1 > 0$, $\mathcal{J}$ is a final boundary. Thus,

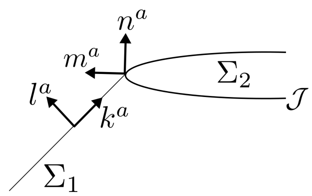
$$\delta I_{\mathcal{J}} = \pm\int_{\mathcal{J}} d^2x\sqrt{\sigma}\alpha_1. \tag{75}$$



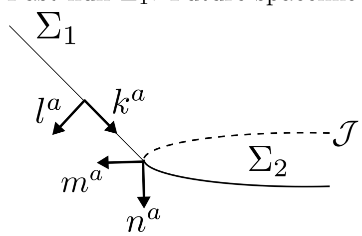
(a) Future null Σ_1. Past spacelike Σ_2.



(b) Past null Σ_1. Future spacelike Σ_2.



(c) Future null Σ_1. Future spacelike Σ_2.



(d) Past null Σ_1. Past spacelike Σ_2.

Figure 3: Examples of null-spacelike joins in $2+1$ Minkowski spacetime showing the orientation of the vectors k^a, l^a, n^a and m^a. The subcaptions illustrate whether a given null or spacelike surface is part of the future or past boundary of $\mathcal{M}$.

Again, using $\delta(n.k) = -\alpha_1(n.k) \Rightarrow \alpha_1 = -\delta(\ln(|n.k|))$ we find that the corner term is

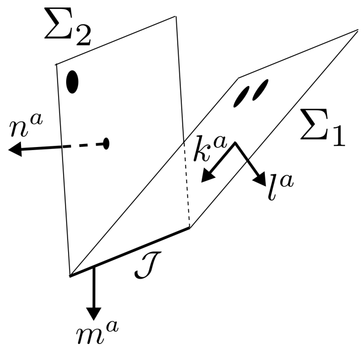
$$I_{\mathcal{J}} = \mp \int d^2x\sqrt{\sigma}\ln(|n.k|). \tag{76}$$

Finally, let us consider the case when Σ_2 is timelike and $\mathcal{J}$ is null. An example of this is shown in Figure 5.

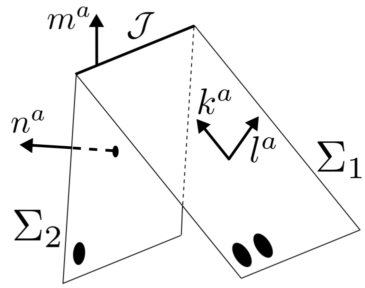
As discussed in Section 4.3 there is no contribution to a non-spatial corner from Σ_1, and the contribution from Σ_2 is given by Eqn (55). Moreover, the normal j^a to $\mathcal{J}$ coincides with that of Σ_1, i.e., $j^a = k^a$. Choosing the spatial basis $\{\tilde{s}^a, \hat{s}^a\}$ on Σ_1 such that $\tilde{s}^a$ is in $\mathrm{T}\mathcal{J}$ and noting that $n^a k_a = 0$, $n^a = w_1\hat{s}^a$, with $n.n = 1 \Rightarrow w_1^2 = 1$. If we expand $t^a = u_2k^a + v_2l^a + w_2\hat{s}^a + z_2\tilde{s}^a$, $n^a t_a = 0 \Rightarrow w_2 = 0$. The variations of the metric

$$\begin{aligned} \delta g_1^{ab} &= k^{(a}(\alpha_1 l^{b)} + \beta_1 k^{b)} + s_1^{b)}) \\ \delta g_2^{ab} &= n^{(a}(\alpha_2 n^{b)} + t^{b)}). \end{aligned} \tag{77}$$

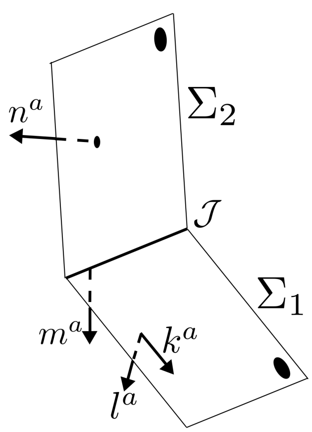
Expanding $s_1^a = \gamma_1\hat{s}^a + \gamma_2\tilde{s}^a$, the matching condition Eqn (68) implies that all the variables in the variation except for u_2 and γ_1 which are related by $w_1u_2 = \gamma_1$, vanish on $\mathcal{J}$. Since $t^a j_a = -v_2 = 0$ there is no corner term contribution. This is consistent with the fact that the inner product of the normals $k.n = 0$ and that $\delta(n.k) = \alpha_2(k.n) = 0$.



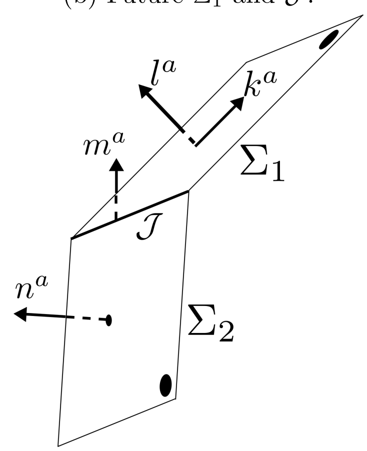
(a) Past Σ_1 and $\mathcal{J}$.



(b) Future Σ_1 and $\mathcal{J}$.

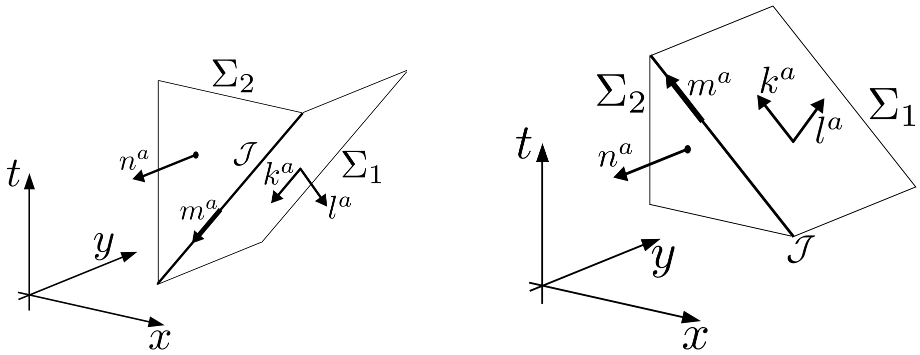


(c) Past Σ_1 and $\mathcal{J}$.



(d) Future Σ_1 and $\mathcal{J}$.

Figure 4: Examples of the null-timelike case with a spacelike join in $2+1$ Minkowski spacetime. In each example we show a portion of the null and timelike surfaces Σ_1 and Σ_2 respectively. One (Two) dot(s) on a surface indicates that, from the perspective of the diagram, you are seeing the inside (outside) face of the surface with respect the region $\mathcal{M}$ that it bounds. The subcaptions illustrate whether the null surface and join are part of the future or past boundary of $\mathcal{M}$.



(a) Past Σ_1 and $\mathcal{J}$.

(b) Future Σ_1 and $\mathcal{J}$.

Figure 5: Examples of the null-timelike case with a null join in $2+1$ Minkowski spacetime. In each example we show a portion of the null and timelike surfaces Σ_1 and Σ_2 respectively. From the perspective of the diagram the outside faces of the surfaces can be seen, with respect the region $\mathcal{M}$ that they bound. The subcaptions illustrate whether the null surface and join are part of the future or past boundary of $\mathcal{M}$.

4.5 Reparametrisation and the Null Boundary Action

Combining all the boundary terms we find that

$$I_B = \sum_i I_{K_i} - \sum_j I_{\kappa_j} + \sum_k I_{\mathcal{J}_k}, \tag{78}$$

where i, j, k range over the number of non-null boundary components, the number of null boundary components and the number of corners, respectively. The null boundary term Eqn (61) is not invariant under reparametrisation. Let us consider the reparametrisation of the null vector

$$\tilde{k}^a = f(\lambda, x) k^a. \tag{79}$$

where f is strictly positive and x is a local coordinate on the null generators. The surface gravity associated with $\tilde{k}^a$ then transforms as

$$\tilde{\kappa} = f(\lambda, x)\kappa - \frac{df}{d\lambda}, \tag{80}$$

so that

$$\begin{aligned} I_{\tilde{\kappa}} &= \int_\Sigma d^2x\sqrt{\sigma}(d\tilde{\lambda}\tilde{\kappa}) \\ &= \int_\Sigma d^2x\sqrt{\sigma}(d\lambda\kappa) - \int_\Sigma d^2x\sqrt{\sigma}(\frac{d\ln f(\lambda, x)}{d\lambda}) \\ &= I_\kappa - \int_{\mathcal{J}_f} d^2x\sqrt{\sigma}\ln f(\lambda_f, x) + \int_{\mathcal{J}_i} d^2x\sqrt{\sigma}\ln f(\lambda_i, x) + \int_\Sigma d^2x d\lambda\frac{d\sqrt{\sigma}}{d\lambda}[\ln f(\lambda, x)]. \end{aligned} \tag{81}$$

The second and third terms exactly cancel the corner contribution (Equations(70), (76))

$$\mp\int_{\mathcal{J}} d^2x\sqrt{\sigma}\ln(|\tilde{k}.n|) = \mp\int_{\mathcal{J}} d^2x\sqrt{\sigma}\ln(|k.n|) \mp \int_{\mathcal{J}} d^2x\sqrt{\sigma}\ln f(\lambda, x), \tag{82}$$

which is negative or positive depending on whether $\lambda|_{\mathcal{J}}$ is an final or initial value. Here n^a represents the normal to the "other" surface at the join $\mathcal{J}$ which can be either null or non-null. The presence of the last term in Eqn (81), which can be rewritten as

$$\Delta I_B = \int_\Sigma d^2x\sqrt{\sigma}\Theta(\lambda, x)d\lambda \ln f(\lambda, x) \tag{83}$$

shows that the boundary action is *not* invariant under reparametrisation. Let us now interpret this. Let us note first that under allowed variations, (those that hold the boundary geometry fixed) the *variation* of ΔI_B vanishes, since it depends only on the boundary geometry. Thus the variation of the boundary action is reparametrisation invariant although the action itself is not. As a general rule, it is differences in the action that are important. Presumably, we can assume this to be true in quantum gravity as well.

Recall the discussion in section 3.1, where we noted that the boundary action is not gauge invariant under general gauge transformations although its *variation* is. This is exactly what is happening here. The surface gravity κ is a component of a connection and (as seen in Eqn (80)) transforms inhomogeneously under gauge transformations. Reparametrisation changes the "size" of the null normal k and is therefore not in the little group. The behaviour of the boundary action under reparametrisation is an example of the general phenomenon discussed there.

We also clarify that this lack of reparametrisation invariance of the null boundary action does not result in any arbitrariness in physical quantities. This is because what appears in physical quantities is the *difference* of two connections, which is a gauge covariant quantity. This point is explained further in the conclusion.

If one wishes, one could add "counter terms" to the boundary action to render it reparametrisation invariant. For example

$$-\int_\Sigma d^2x\sqrt{\sigma}d\lambda[\Theta(\lambda, x)\ln\frac{d\lambda}{dt}], \tag{84}$$

with t being an arbitrary affine parameter, does the job. Another possibility[1] is

$$-\int_\Sigma d^2x\sqrt{\sigma}d\lambda[\Theta\ln\Theta]. \tag{85}$$

A third possibility is

$$-1/2\int_\Sigma d^2x\sqrt{\sigma}d\lambda[\Theta\ln s_{ab}s^{ab}], \tag{86}$$

where s_{ab} is the shear tensor of the null geodesic congruence ruling the null surface. Of these, the first Eqn (84) depends on a choice of affine parametrisation, which brings in some arbitrariness, since the parameter t can be rescaled by $t \to c(x)t$, where $c(x)$ depends on the null generator. A more serious problem is that including this counterterm spoils the additivity of the action for regions separated by a null boundary. For, the notion of an "affine" parameter in general will depend on which region we use to define the affine parameter. The two counter terms will therefore differ in value and therefore spoil the additivity of the action, which was one of our prime motivations.

The second and third Eqn (85,86) do not suffer from this ambiguity. However, they too have a problem: the counterterm is not differentiable if the expansion or shear vanishes. Our view is that there no real need to add a counterterm at all since the lack or reparametrisation invariance does not manifest itself in physical quantities.

5 Conclusion

The main new advance of this paper is the realisation that the tetrad formulation of Einstein's theory permits a unified approach to boundaries of all signatures. The calculations are considerably simplified and the use of differential forms permits us to integrate over boundary manifolds regardless of their signature. Our derivation of the corner terms too is extremely simple. Our methods are complementary to [11, 12, 1] and our perpective is somewhat different. The differential form version of the boundary term also makes it obvious that the boundary corrected action is additive. In any splitting of a spacetime into pieces, the boundary term I_B Eqn (13) appears twice on the shared boundary with opposite orientation and so cancel out. The gauge non invariance of the boundary action does not affect us here since the difference of the two connections is a gauge covariant object. In particular, the reparametrisation non invariance of the null boundary action does not spoil the additivity of the action.

In this paper we have worked within the Dirichlet formalism for gravity in which the pullback metric q_{ab} is held fixed on the boundary during the variation. One can also conceive of "Neuman gravity" in which the conjugate variable is held fixed. For example if the boundary is spacelike, the quantity $\sqrt{q}(K^{ab} - 1/2Kq^{ab})$ related to the extrinsic curvature is conjugate to the three-metric. There has been recent work [19] exploring this possibility, albeit in the Euclidean context. Such alternate formalisms are of interest since it is far from clear which ensemble would prove the most advantageous in quantisation. It is also possible that these different choices may lead to different quantum theories. For example, it is known in statistical mechanics that conjugate ensembles may not always be equivalent. Such issues are particularly acute in the case of long range forces like gravity. A classic example is the stability question of a black hole in equilibrium with thermal radiation in a box.

A notable feature of the boundary term Eqn (13) is that it is not gauge invariant although its *variation* is. One must bear in mind that the boundary action is only determined up to a functional of the boundary data that is held fixed, in our case the pullback of the metric to the boundary. One may worry that the value of the action changes under change of gauge. However, there is no cause for concern. In a path integral formulation observable quantities are related to the absolute value squared of the Feynman amplitude in Eqn (1). This leads to a closed time path integral of the Schwinger-Keldysh formalism. The quantity that appears in the exponent is now $S(X_3, \Gamma) - S(X_3, \overline{\Gamma})$, where Γ and $\overline{\Gamma}$ are histories going between X_1 and X_3. While the two histories share the same final geometry X_3, they have different values of the connection at the final point. The two boundary terms at X_3 then combine to give a gauge invariant answer, since the *difference* of two connections transforms homogenously. Another situation that arises is when one considers asymptotically flat spacetimes, takes the boundary to infinity and interprets the boundary term in terms of the total mass. In this case as is well known, we need to make a background subtraction in order to get a finite answer. Once again, this subtraction results in a gauge invariant boundary term, since the difference of two connections is a gauge covariant object. The gauge non invariance of the boundary term is precisely what we have exploited in order to identify the corner terms. This remark has a parallel in the metric formulation too. The integrand in the boundary term Eqn (45) is also not coordinate invariant since it depends on the affine connection. The general allowed variation of the metric Eqn (37) can (at points of the boundary) be interpreted as a diffeomorphism generated by the vector field $\xi_a = \phi Q_a$, where ϕ is any function that vanishes on the boundary.

Under such a diffeomorphism, the integrand in the boundary term changes by a total derivative and this permits us to identify the corner terms in the metric formulation.

In the literature, it is suggested that the corner terms [15] or their close analogs [13] may pick up imaginary contributions. (Imaginary contributions figure heavily in the Lorentzian Gauss-Bonnet theorem as well.) Using our methods, such contributions would not be detected, as they have zero variation. However, the origin of such terms can be understood when the normal changes from timelike to spacelike. We have chosen different adapted frames depending on whether the normal to the boundary is null, spacelike or timelike. This is because no Lorentz transformation can connect these different normals. However, in connecting spacelike normals to timelike normals, it is possible to use complex Lorentz transformations. If we complexify the Lorentz group to $O(2,\mathbb{C})$, the element $\Lambda =$

$$\begin{pmatrix} \cosh(\eta + i\pi/2) & \sinh(\eta + i\pi/2) \\ \sinh(\eta + i\pi/2) & \cosh(\eta + i\pi/2) \end{pmatrix}$$

which has complex rapidity, $\eta + i\pi/2$ does the job of connecting spacelike and timelike normals. Thus every time the normal crosses a null direction, (crossing counted with sign), the action picks up an imaginary contribution $i\pi/2 \int dA$. This imaginary area term has been interpreted as black hole entropy by Neiman and we refer the reader to [15] for a fuller discussion. While such a term affects the *value* of the Action, it does not affect the *variation*, since the variation of the area vanishes. Note however, that no Lorentz transformation (real or complex) can relate a null normal to a spacelike or timelike one. It seems necessary to use different canonical forms for null and non-null normals.

The case of null boundaries has not received much attention till the recent works of Neimann[15, 16, 17, 18], Parattu et al [11, 12] and Lehner et al [1]. Neimann was mainly interested in imaginary contributions to the action at the join of null boundaries. He used affine parametrisations to describe the null generators, which is unnecessarily restrictive in the present context. The treatment of Parattu et al [11] allows for arbitrary parametrisation of the null generators and correctly identifies the form of the boundary action for null surfaces. However, these authors do not consider the corner terms, which are necessary for a complete treatment of the boundary action. In a second paper [12], they attempt a unified description of both the null and non null case. Their treatment is coordinate bound and makes assumptions about the behaviour of the normal away from the boundary. Lehner et al [1] provide a metric treatment of the null boundary terms and identify the corner terms. They also have a detailed discussion of reparametrisation invariance and suggest counterterms to be added to the boundary action.

In the present work, we use the power of Cartan's tetrad formulation and differential forms to considerably simplify the treatment. Differential forms give us a unified approach to boundaries of all signatures. We compute the corner terms quite simply using the local Lorentz invariance of the tetrad formalism. In the mathematical section we also give a classification of all possible corner signatures, including the case of null joins (see Figure 5) that have not been considered in the above works. In order to reach a wider audience we also translate our results into the metric language which is more familiar to readers. We have also noted the contribution which come from "creases" that appear in spacetimes with a dynamically evolving black hole exterior. Finally, we offer a perspective on reparametrisation invariance (RI) in the null case, which differs slightly from Ref [1]. Rather than try to restore RI, we note that the lack of RI in the boundary action does not affect any physical quantity in the path integral.

We close with a remark regarding the asymptotics of gravitational fields. Let us compare the value of the action in the second order Einstein-Hilbert form and the first order form. For asymptotically flat spacetimes, the metric tends to its flat asymptotic form g_0 at the rate $(g-g_0) = O(1/r)$. As a result, the difference between the connection Γ of g and the flat connection Γ_0, $\Delta\Gamma = \Gamma - \Gamma_0$ goes as $\Delta\Gamma = O(1/r^2)$ and $R = O(1/r^3)$. The Einstein Hilbert form diverges logarithmically at radial infinity ($\int Rr^2 dr \approx \int dr/r$) but the first order form converges: ($\int(\Delta\Gamma)^2 r^2 dr \approx \int dr/r^2$). This allows an interpretation of the 4-momentum as a well defined variation of the action, i.e as a Noether charge. While there has been much work on null infinity[20], we are not aware of any discussion of boundary counterterms in this context, for example, in the derivation of the Bondi mass. The issue of null boundaries has been neglected until the recent interest generated by [11, 12]. There has been recent work [21, 22] reviving the topic of asymptotic null infinity [23, 24, 25] and relating it to soft theorems in particle physics. We hope that our treatment of null boundaries may help understand null asymptotics of gravitational fields.

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