What determines the non-equilibrium rate of profit?*

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Abstract

Taking as our hypothesis a form of the labour theory of value, and *without assuming equilibrium*, we derive an equation that yields the profit-rate π as a function of time t. For a mature economy, $\pi(t)$ reduces to the product of two factors: (i) a certain *retarded average* of the sum of the growth-rates of labour productivity and of employment measured by total hours worked, and (ii) the ratio of the current rate of surplus value to its own retarded average. We also suggest an empirical test of the equation.

Keywords and phrases: labour-value, profit-rate, time-dependence, non-equilibrium economics

I. Introduction

Any model of capitalist dynamics that rests on an assumption of economic equilibrium can have at best a limited explanatory value. Even without having witnessed the upheavals of the past fifteen years, I doubt that any serious observer would want to deny that *disequilibrium* appears more characteristic of capitalist economies than its opposite. The history

* The present article is a revised version of a 1982 manuscript that was submitted unsuccessfully to various economics journals. The main equations are the same as before, with some improvements and changes of notation. The surrounding discussion has been revised significantly, especially the concluding section.

of capitalism presents us not with a smooth development, but with a series of booms and subsequent "crises" interspersed with even more disruptive episodes like wars, including both wars fought out among the nations of the capitalist core and wars inflicted by the core on the "halo regions" (the colonies and neo-colonies) of the capitalist world. Faced with this chaotic panorama, we must ask whether there exist economic models robust enough to survive under conditions where "competitive equilibrium", if it ever existed, has been lost.

As soon as one decides to seek such a model one comes face to face with a problem. How are we to attribute exchange-values to commodities? Neo-Ricardian approaches assign values-qua-prices by reference to an input-output matrix or its generalizations. Other approaches, even farther removed from reality, appeal to the hypothetical "utilities" and "preferences" of atomized economic actors. In both cases, an appeal to uniform growth or to some other type of equilibrium is essential. One might attempt to rescue such models by adding in small fluctuations away from equilibrium, but that could be done only if the equilibrium could be claimed to be stable, or if it could be claimed to become stable via some sort of coarse-graining. As far as I know, such claims have not been put forward with any seriousness, and no such generalization has been attempted.

In the face of such difficulties, it is tempting to pursue the long-standing idea that exchange-value derives from labour-value. According to this hypothesis, known usually as the "labour theory of value", prices are merely a phenomenal form taken by values which are more fundamentally defined by asking how much accumulated human labour went into producing a given item. One can of course raise many questions about how well defined this labour-value really is, how it relates to prices, etc. This is not a discussion I want to enter into here. Instead I simply want to demonstrate that if one uses labour-time as one's measure of value (denominated in hours, for example), and if one defines profit in terms of labour-values, then one can derive a general equation for profit-rate that expresses it as the historical resultant of variables that are relatively close to being directly observable, namely the rates of growth of productivity and of the labour-force, together with the rate of surplus value (or equivalently exploitation). †

If one accepts the labour-value hypothesis, then one would expect that profit defined thereby would, at a minimum, offer a conceptual tool useful for analysing the processes underlying capitalist markets. One might also go further and expect that this labour-value profit would equate roughly to profit as measured by prices. If so then the underlying assumptions would be open to an empirical test.

II. Assumptions

The analysis to be presented herein will be self-contained, but it may be helpful to point out its relation to the discussion of the "falling rate of profit" in volume III of Das Kapital, chapters 13-15 (which seems to be Engels' working out of a preliminary draft by Marx) [3].

Working entirely with labour-values, the analysis presented there argues that the rising productivity of labour-power occasioned by technological progress causes the profit-rate to fall, because it increases the ratio of constant to variable capital. If one tries to paraphrase the reasoning there, but if in doing so one shifts the emphasis from investment in technology to investment *as such*, then one comes up with the following:

Profit, insofar as it is invested, implies an increase in the mass of constant capital, which in turn raises the ratio of total capital to surplus value. But, since only surplus

† The formula thus generalizes the well known equilibrium result that profit-rate in essence just measures the rate of growth of the economy. See for example the Sraffa input-output model in [1] or the von Neumann "balanced growth" ray in [2].

value ends up as profit, the rate of profit must fall secularly. In a word, present investment diminishes future profit-rates.

It seems that the analysis in [3] was put forward at a time when a falling rate of profit was (or was at least perceived by contemporary writers on economics to be) an observed fact. The question was not *whether* profit-rates would fall, but *why* they were falling, and the analysis in [3] was meant to be the explanation. Of course, such an explanation entails also a prediction, but only to the extent that the falling tendency prevails over certain "counteracting causes" [Entgegenwirkende Ursachen] which are identified in [3], including: the lengthening of the working day; an increase in the rate of exploitation; the cheapening of capital due to rising productivity; and the effects of foreign trade and imperialist super-profits (to use a more recent term). If these causes are present, they will tend to raise the profit-rate, and whether the falling or rising tendency will prevail cannot be decided without a more quantitative analysis.

In what follows, and within a specific, idealized framework, we will carry out such an analysis. We will find that the profit rate will indeed fall if it begins at a high enough value (cf. equation (6) and Figure 1). But we will also see that this is not the full story. Taking into account the first three of the counteracting causes, and allowing also for possible growth in the number of workers, we will derive an equation that yields the profit-rate $\pi(t)$ as a determinate function of time t. According to this equation, $\pi(t)$ in a mature economy is known if the "counteracting causes" are known quantitatively at all earlier times, up to and including t. A significant (and intuitively plausible) result of our analysis will be that $\pi(t)$ will tend to evolve toward the underlying rate of growth of the real economy. In particular, this underlying growth-rate will act as a ceiling above which, except for brief fluctuations, the profit-rate cannot rise.

Our analysis will treat the economy as a whole, that is we will treat everything *in aggregate*. Our main results will be equations (5), (16), and (18). In deriving them, we will employ for simplicity a *continuous-time* model of the economy. There will not be "annual outlays" but instead *flows* of raw materials, of labour, etc. We will also work in the limit of zero "latency time", whence the only capital that will enter into the profit-rate will be fixed capital. (With nonzero latency time, other forms of capital like inventory and "advanced" wages would also influence the profit-rate, but some of them could still be treated as special cases of fixed capital.) Thus we will make (at least) the following assumptions and idealizations.

(1) A *closed* capitalist economy. No (or only equal) trade, no destruction of capital (as by war).

(2) The rate of profit is equal to total net profit divided by total fixed capital.

(3) Profit and capital will be measured by their labour-values, as will all other items.

(4) All profits are invested and there is no saving by workers.

(5) All investment goes into capital formation.

III. First consequences

Let all values be measured in current labour time (replacement values at time t) and let $K = K(t)$ be the (value of the) total fixed capital at time t. (Recall that we are treating the economy in aggregate.) If $\pi = \pi(t)$ is the *rate* of profit then in unit time the *amount* or "mass" of profit will be by definition πK, whose reinvestment causes the value of fixed capital to *increase* at a rate $dK/dt = \pi K$. At the same time however, the cheapening of capital goods *reduces* K at the rate $dK/dt = -rK$, where r is the rate of productivity growth in the capital goods sector. Combining these competing effects yields

$$\frac{dK}{dt} = -rK + \pi K = (\pi - r)K \tag{1}$$

If only a fraction $\phi = \phi(t)$ of profit ends up being invested then (1) becomes

$$\frac{dK}{dt} = -rK + \phi\pi K \ , \tag{1a}$$

but for simplicity we are taking $\phi \equiv 1$ (assumption 4 above). We are also ignoring any destruction of capital, as by war or by scrapping still functional machinery for lack of demand (assumption 1 above).

To proceed further, define P as the number of workers, λ as the fraction of the day each spends working on average, and w as the wage rate (hours of wage paid to a worker per hour of his/her work). Recall here that wages are being measured directly in units of labour-time. Accordingly, w and λ are both dimensionless numbers between 0 and 1.* The total wages paid per unit time are thus $w\lambda P$, which subtracted from the value produced, namely λP, leaves for the collective capitalist a surplus value s of

$$s = \lambda P - w\lambda P \tag{2}$$

It is convenient to combine λ and P into the product $L = \lambda P$ (which is the aggregate labour-time expended per unit time) and to define $\sigma = 1 - w$, which will be called the *rate of surplus value*, and which is also a pure number between 0 and 1 representing the fraction

* We neglect those rare circumstances in which w might exceed unity or fall below zero. We similarly neglect the possibility (if it is one) that "speedup" and overtime become so extreme that λ effectively exceeds unity.

of the workers' production retained by the capitalists. † Combining these definitions yields for the mass of surplus per unit time,

$$s = (1 - w)\lambda P = \sigma L$$

Finally, since by definition mass of profit equals s, we find for the *rate* of profit, $\pi = s/K$, simply

$$\pi \;=\; \text{rate of profit} \;=\; \frac{\sigma L}{K} \tag{3}$$

Now how does this labour-value profit-rate change with time?

To answer this question, we need only differentiate (3) and combine the result with (1) to obtain

$$\frac{\dot{\pi}}{\pi} \;=\; \frac{\dot{\sigma}}{\sigma} + \frac{\dot{L}}{L} - \frac{\dot{K}}{K} \;=\; \frac{\dot{\sigma}}{\sigma} + \frac{\dot{L}}{L} + (r - \pi) \;,$$

an equation which will look slightly simpler if we define

$$\beta \;=\; \frac{\dot{L}}{L} + r \tag{4}$$

so that it becomes

$$\frac{\dot{\pi}}{\pi} = \frac{\dot{\sigma}}{\sigma} + \beta - \pi \tag{5}$$

Notice here that $\frac{\dot{L}}{L} = \frac{\dot{\lambda}}{\lambda} + \frac{\dot{P}}{P}$ is just the rate of growth of the number of workers plus the rate of lengthening of the working day. Hence (as is anyway obvious) $L = \lambda P$ can rise over long periods only if P does, since λ is bounded above by unity.

One sees how almost all of the ingredients of the "falling profit-rate" discussion are represented in (5). If one were to keep only the third term in (5), it would reduce to $\dot{\pi} = -\pi^2$, and one would deduce that π must decrease forever, falling asymptotically to zero (or diverging to $-\infty$ if it were negative). But this would be to neglect the effect of the

† The commonly defined "rate of exploitation" e is related to σ by $e = \sigma/(1 - \sigma)$.

two "counteracting" terms. The first of these, namely $\dot{\sigma}/\sigma$, can check the fall to the extent that the rate of exploitation increases, but its effect necessarily dies out as σ approaches its maximum value of unity. The other counteracting term, namely $\beta = \frac{\dot{L}}{L} + r = \frac{\dot{\lambda}}{\lambda} + \frac{\dot{P}}{P} + r$, receives three distinct contributions. The contribution $\dot{\lambda}/\lambda$ corresponds to lengthening the working day, but its effect is limited in the same manner as is that of $\dot{\sigma}$. Therefore, if β is to be effective for more than a limited time, it can only be because either P (the size of the workforce) or the productivity of labour (corresponding to r) grows indefinitely. The net effect is that the long-term profit rate is governed by these two underlying growth rates. (Arguably they cannot act forever either, because no exponential growth can last forever. But the time-scales on which these limitations would assert themselves are evidently much longer than those belonging to σ and λ.)

In the next section, we will treat equation (5) in full generality, and we will find that from it, we can deduce the profit-rate at any time t, provided that we are given the values of σ and β at all earlier times.[♭] For now, we just note that, if we ignore the transient effects of a changing σ, the profit-rate π necessarily moves toward β, decreasing when it exceeds β, but increasing if it falls below β. In this sense β is the "reference value" that π always seeks.

But before turning to the general analysis just referred to, let us consider a simple example of what can be expected. Suppose that both σ and β are independent of time and that at $t = 0$ there is some labour but no capital in existence ("new economy"). Integration of equation (5) (conveniently done by changing to the variable $z = 1/\pi$) yields in this situation

$$\pi(t) = \frac{\beta}{1 - e^{-\beta t}} \ . \tag{6}$$

[♭] If the economy is insufficiently mature, then one needs also the initial value of π or its equivalent.

As illustrated by the upper curve in Figure 1, the profit-rate under these conditions falls from an infinite initial value to its equilibrium value of β on a time-scale whose duration, namely β^{-1}, is itself set by β, i.e. by the underlying rate of growth of productive capacity. The "falling tendency" and the "counteracting causes" are then in balance.

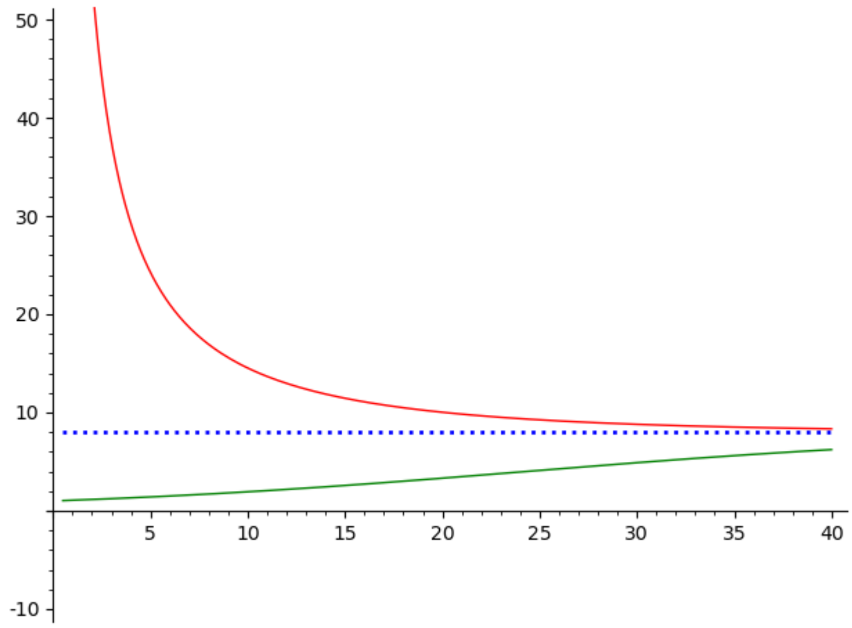

Figure 1. Profit-rate π (in percent) vs. time (in years), under the assumption that β and σ in equation (5) are constant in time, and that $\beta = 8\%$ (corresponding for example to an annual rate of population-growth of 3% and a rate of productivity-increase of 5%). The **falling curve** plots the solution of (5) for which zero capital is present at $t = 0$ (perhaps as the result of a particularly destructive war). Initially infinite, $\pi(t)$ falls rapidly toward its steady-state value β (dotted horizontal line). The **rising curve** plots the solution for which the initial profit-rate is only 1% (perhaps as the result of an "overaccumulation" of capital). In this case π approaches β from below, albeit not as quickly. Although the rate of surplus-value has been assumed to be independent of time ($\dot{\sigma} = 0$), its actual value need not be specified. Provided that the initial profit-rate $\pi(0)$ is given, $\pi(t)$ is determined by (5) for all subsequent times.

IV. The profit rate more generally

In general neither β nor σ will be constant. Nevertheless, it is clear from the fact that (5) is a first-order differential equation that given the initial value of π, we can, within the limits of our simplified model, deduce $\pi(t)$ for all t if we know the history of β together with that of σ. By working out this dependence explicitly, we will arrive at the general solution for $\pi(t)$ recorded in equation (16). In consequence of this equation, we will also see that the initial conditions tend to drop out in a "mature" economy, leading to the result that the profit-rate at a given time is the product of two averages which summarize the historically recent values of β and σ, respectively. We will also derive an upper bound on π that is entirely independent of initial conditions, and is relatively independent of σ as well.

Our task, then, is to solve (5), presupposing that both β and σ are known functions of time. As it stands, (5) contains a term quadratic in π, but we can render it linear by working with $1/\pi$ rather than π itself. In fact the new variable Z defined by

$$Z \;=\; \sigma/\pi \tag{7}$$

will be a slightly more helpful choice. From equation (3) we see that $Z = K/L$ measures the fixed capital per worker hour. It is thus closely related to what is sometimes called the "organic composition of capital". Rewritten in terms of Z equation (5) becomes simply

$$\frac{dZ}{dt} + \beta\, Z \;=\; \sigma \tag{8}$$

Thus β acts like a "restoring force" that (when positive) pulls Z toward 0, while σ acts like a "source" that pushes Z toward higher values. (More exploitation implies more rapid accumulation of surplus value.) At any rate, we have in (8) a linear equation whose general solution is

$$Z(t) = Z_0\; e^{-B(t,0)} \;+\; \int_0^t e^{-B(t,s)}\;\sigma(s)\; ds \tag{9}$$

where

$$B(t_2, t_1) = \int_{t_1}^{t_2} \beta(t)\ dt \tag{10}$$

and $Z_0 = Z(t = 0)$ is the initial value of Z. (One can verify this solution by direct substitution into (8).) From $Z(t)$ we can recover $\pi(t)$ trivially, but first let us interpret the expression $B(t_2, t_1)$ and certain integrals in which it occurs.

To that end, notice first that since e^{-B} is always positive the integral in (9) constitutes a weighted sum of σ with recent values counting most heavily (unless the economy is shrinking!) and very early values being exponentially damped. Appropriately normalized this integral therefore defines a certain kind of "moving average" of σ (or of any given function of time) which I'll denote by $\langle \cdot \rangle$:

$$\langle\, \sigma(t)\, \rangle \;=\; \int_0^t e^{-B(t,s)}\ \sigma(s)\ ds \Big/ \int_0^t e^{-B(t,s)}\ ds \tag{11}$$

The denominator of (11) can also (though less obviously) be interpreted as an average, this time of β itself. To that end let us define $\bar\beta = \bar\beta(t)$ through the equation

$$\frac{1 - e^{-t\bar\beta(t)}}{\bar\beta(t)} = \int_0^t e^{-B(t,s)}\ ds\ . \tag{12}$$

Then $\bar\beta$ is a kind of "self-weighted moving average" of β in the following sense (see Appendix):

(1) For each t, equation (12) defines $\bar\beta(t)$ uniquely.

(2) If β is the constant function β_0 then $\bar\beta = \beta_0$.

(3) If (for all t) $\beta_1 \le \beta_2$ then $\bar\beta_1 \le \bar\beta_2$; in particular any (lower or upper) bound for β is also one for $\bar\beta$.

Moreover this second average is closely related to the first one, since, as proven in the appendix,

$$\frac{\bar{\beta}}{1-e^{-t\bar{\beta}}} \;=\; \frac{\langle\,\beta\,\rangle}{1-e^{-B(t,0)}} \tag{13}$$

In order to bring out further the intuitive meaning of the weighting factor e^{-B} which defines the average $\langle\,\cdot\,\rangle$, let us observe that according to (10) and (4), $B(t_2,t_1)$ is the sum of the integrated growth-rates of L and of productivity. If we temporarily introduce a symbol Π to represent productivity and define $Q = L\,\Pi$ then $\beta = \dot{L}/L + \dot{\Pi}/\Pi$ is the rate of change of $\log Q$ and its integral B is just the logarithm of $Q(t_2)/Q(t_1)$. Thus

$$e^{B(t,s)} = Q(t)/Q(s) \tag{14}$$

is a kind of measure of how much the productive capacity of the economy has grown between t_1 and t_2, and the effect of our weighting factor $e^{-B(t,s)}$ in the above integrals is to discount the contribution from a given historical moment in proportion as the productive capacity then was smaller. In a growing economy, such an average damps out the contribution from early times.

Thanks to these observations, we can also express the average (11) as

$$\langle\,\sigma(t)\,\rangle \;=\; \int_0^t Q(s)\;\sigma(s)\;ds \Bigg/ \int_0^t Q(s)\;ds \tag{11a}$$

while the corresponding recasting of $\bar{\beta}$ appears as

$$\frac{1-e^{-t\,\bar{\beta}(t)}}{\bar{\beta}(t)} \;=\; \int_0^t Q(s)\;ds \;/\; Q(t)\;. \tag{12a}$$

With these definitions in mind, we can assemble equations (9), (14), (11a), and (12a) to obtain [where $Z = Z(t)$, $Q = Q(t)$, $\bar{\beta} = \bar{\beta}(t)$]

$$Z = \frac{Z_0 Q_0}{Q} + \langle\,\sigma\,\rangle\frac{1-e^{-\bar{\beta}t}}{\bar{\beta}}$$

which, with the aid of (14) and the identity (13), becomes

$$Z = \frac{Z_0 Q_0}{Q} + \langle\,\sigma\,\rangle \frac{1 - Q_0/Q}{\langle\,\beta\,\rangle}$$

or after rearrangement,

$$Z \;=\; \frac{\langle\,\sigma\,\rangle}{\langle\,\beta\,\rangle} \;+\; \frac{Q_0}{Q}\left[Z_0 - \frac{\langle\,\sigma\,\rangle}{\langle\,\beta\,\rangle}\right] \tag{15}$$

This is our main result expressed in terms of $Z = \sigma/\pi$. In a sufficiently mature economy (one which has already expanded by a large factor) Q_0/Q will be small compared to unity and we will be left with the approximate equality, $Z = \langle\,\sigma\,\rangle / \langle\,\beta\,\rangle$.

Finally, we can substitute the definition (7) of Z to obtain an equation for the profit-rate at time t which, after a small rearrangement, furnishes our main equation for π:

$$\pi \;=\; \frac{\langle\,\beta\,\rangle\;\sigma/\langle\,\sigma\,\rangle}{1 + \frac{Q_0}{Q}\left[\frac{\langle\,\beta\,\rangle}{\pi_0}\frac{\sigma_0}{\langle\,\sigma\,\rangle} - 1\right]} \tag{16}$$

Once again the term containing Q_0/Q can be dropped in a mature economy and we are left in that case with the approximate equality

$$\pi \;=\; \langle\,\beta\,\rangle\;\frac{\sigma}{\langle\,\sigma\,\rangle} \tag{17}$$

Even without making any assumption about maturity, we deduce from (16) the inequality

$$\pi \;\le\; \langle\,\beta\,\rangle\;\frac{\sigma}{\langle\,\sigma\,\rangle}\;(1 - Q_0/Q)^{-1} \tag{18}$$

which bounds the profit-rate above by an expression not much bigger than (17), the ratio being given by $1/(1 - Q_0/Q)$, a factor close to unity for which $(1 + Q_0/Q)$ should provide an adequate approximation in any but a very young economy.

What the last three equations tell is that, apart from initial transients and temporary fluctuations, the profit-rate coincides with the growth-rate β averaged over historically recent times. They also tell us that the fluctuations about this average come from the factor $\sigma/\langle\,\sigma\,\rangle$ which will be appreciable only when σ (the rate of surplus value) rises sharply above, or falls sharply below, its historically recent average $\langle\,\sigma\,\rangle$. And since σ is in any case less than 1, a sharp increase is possible only if it was recently much less than unity.

V. Comments, Extensions, and Possible Tests

Under conditions of equilibrium, the underlying growth-rate is the only time-scale in play, and one would expect to recognize this same rate no matter what variable one chose to monitor, be it the profit-rate or anything else. It is therefore unsurprising that equilibrium models ("neo-Ricardian" and others) produce the result $\pi = \beta$. In disequilibrium conditions, however, some variables will be increasing while others decrease, fluctuations will be significant, if not dominant, and the reasoning behind the equilibrium results will fail. It is thus interesting that, at least in the simple model of this paper, the profit-rate is still governed to a large extent by the growth rates, r and $\dot{L}/L$ (β being their sum). The difference however is that the contemporaneous value of π is no longer tied to the contemporaneous value of β, but to its historical average $\langle\,\beta\,\rangle$ taken in a precisely defined sense. Moreover, significant short-term fluctuations in π will in general take place, depending on the contemporaneous rate of surplus value σ in relation to *its* historical average $\langle\,\sigma\,\rangle$. In a relatively young economy the initial conditions will be important as well. (One can of course set the initial time whenever one wants. The equations will still hold.)

An illustration

By way of illustrating our conclusions, let us imagine a closed economy whose initial time, $t = 0$, is chosen to be around the end of the last "world war", so that at present $t = 70yr$, and suppose that its "productive capacity" Q has expanded since then by a factor of ten: $Q/Q_0 = 10$. The historical averaging that enters into our equations would then weight recent values of β and σ about 10 times more heavily than values from 70 years ago. The latter could thus still make some difference, but not too much. Let us suppose further that β recently has hovered around 2% per year, having been bigger previously such that currently $\langle \beta \rangle = 0.025/yr$, and that the average rate of surplus value has reached $\langle \sigma \rangle = 0.50$. Let us also suppose that in the past couple of years the capitalists, in a desperate attempt to raise profits, have driven σ to its maximum possible value of unity: $\sigma(t) = 1$ (which of course they could never really attain). The inequality (18) would then allow a present-day profit-rate, reckoned in labour-value terms, of at most

$$0.025/yr \times \frac{1.0}{0.5} \times \frac{1}{1 - 0.1} = 0.056/yr$$

In other words, the annual profit rate could not currently exceed 5.6% even if the workers were made to "live on air".

Moreover, even this excess of π over $\langle \beta \rangle = 2.5\%$ would only be temporary. In fact one can estimate in general that $\pi(t)$ can not exceed $\beta(t)$ by an amount ε for a time much longer than $1/\varepsilon$, or somewhat more precisely, this time multiplied by $\ln(1/\sigma) + \ln(1/(\beta' + \varepsilon))$, β' being the value of β at the end of the period.

Further remarks on equilibrium models

I have already expressed more than once herein, the view that equilibrium models are not to be trusted. Perhaps the most convincing evidence for this is the empirical observation that equilibrium is simply not the historical rule, but one can also give theoretical reasons why a hypothetical equilibrium profit rate is likely to be wrong. For this rate is essentially the maximum possible rate, a statement that is in some sense the content of the Frobenius-Perron theorem, which states in particular that for any set of prices there will always be at least one industry in which profitability is at or below its equilibrium value. In itself this is a fairly weak limitation, but presumably one could show that any growth trajectory far from the equilibrium one would soon encounter severe shortages and hence severe "realization" problems, resulting in markedly lower profits. A fuller analysis of this situation might also be able to elucidate the loss of profit inherent in any *return* to equilibrium after new techniques or products have been introduced. In any case, if the equilibrium rate is an upper bound, it follows that slumps can lower profits below this hypothetical value more than booms can raise them. Thus, even if prices and product-mixes averaged out to their equilibrium values, the (nonlinear) relation between them and the (overall) profit rate would in general cause the latter to deviate systematically from its own equilibrium value. Similarly, any nonlinearity in the response of investors to fluctuating prices and markets would invalidate arguments of the Okishio-Theorem type, that employ a notion of capitalist rationality defined solely with respect to the current *equilibrium* price-vector (as in [4] for example).

Possible extensions

In deriving our main results, equations (5) and (16), we have relied on several assumptions and idealizations. Some of them could easily be relaxed. For example one could accommodate a situation where not all surplus value resulted in capital formation by using equation (1a) in place of (1), with ϕ chosen to take account of dividends and personal consumption by capitalists.

The assumption that capital is not destroyed could also be relaxed by incorporating into the variable r an additive contribution representing the rate of its destruction. Such an r could no longer be interpreted simply as a growth-rate, but mathematically the above analysis would go through as before. Equations like (11a) and (12a) would become incorrect, but re-expressed in terms of $B(t, s)$ via (14), all our final results would remain the same.

Some of our other idealizations would be more difficult to do without, notably the assumption that the economy in question is a closed system. One might also wonder whether (3) was a good approximation in a more service-based economy. Similarly, one might question whether, or how, "fictitious" financial capital ought to be included in the variable K, or how to take into account labour which in some sense is "unproductive". (The examples of security guards, internet trolls, and "quants" working for hedge-funds come to mind.)

Possible tests

Beyond these more "technical" questions are the conceptual questions raised by the use of labour-values. In this paper, we are simply adopting labour-values as our starting point and drawing out the consequences mathematically. The advantages of doing so are first of all that we avoid the equilibrium fiction (if I can call it that), and secondly that one can give at least a fairly clear definition of the crucial parameter r measuring productivity growth. One also has the feeling that labour values tap into something basic about the economy which empirical prices do not.* For this reason π might also have a certain analytical interest in its own right. (A good question is why economists worry so much about "productivity" if labour values are irrelevant!)

In this connection, one might wonder whether some other "universal" measure of value could have been used in place of labour-time. Indeed, the fact that β represents just the growth rate of productive capacity (more precisely of the hypothetical capacity for turning out capital goods if all available labour were devoted to them) suggests that some form of the equations relating π with β might be provable without recourse to any theory of value at all. However any attempt to define β in this way (not to mention σ) encounters the ambiguity inherent in defining growth rates in the face of the continual emergence

* The observation seems relevant here that labour-power is the one commodity which (ignoring slave markets and analogous exceptions) is not produced for sale by capitalists.

of new commodities and techniques and the disappearance of old ones which economic growth entails. The present treatment, based on assigning to each commodity its labour-value, renders the crucial variable β unambiguous to a significant extent. Nevertheless, it seems clear that of the two factors entering into its definition, namely hours worked and productivity, the latter is less well defined than the former.

From an empirical standpoint the question would be whether, and to what extent, the phenomenal profit rate can be identified with the labour-value profit-rate studied in this paper. Some economists seem to think that it can, many others think not. At this stage of economic theory, I believe one must treat either answer simply as a hypothesis that one can adopt (or not) as the basis of further analysis, just as one does with hypotheses in the physical sciences. Personally, I have very little idea how to address this issue theoretically, but our results above provide the beginnings of a way to address it empirically. (Beginnings because some of the simplifications we have made would need to be either corrected for or corroborated.) To the extent that r, π and σ can be observed (L unquestionably can), one could compute (from (16) or (5) together with β and σ) π as a function of time and compare the resulting graph with the data. Or if this test were too difficult because $\sigma(t)$ was too hard to observe reliably given available statistics, one could conversely, as a second-best test, deduce $\sigma(t)$ from (5) together with π and β, and see whether the resulting graph was at least intuitively plausible.

Appendix. Some technical details

In terms of the function

$$F(x) \;\equiv\; \frac{1-e^{-x}}{x} \;=\; \int_0^1 e^{-xs} ds \ , \tag{A1}$$

the definition (12) of $\bar{\beta}$ reads

$$F(\bar{\beta}) = \int_0^t e^{-B(t,s)} ds \ . \tag{A2}$$

We want to show that $\bar{\beta}$ is thereby well-defined and also to establish the further assertions made in the text.

In the first place notice that, because the integrand in (A1) is a monotone decreasing function of x, F is also monotone decreasing. Moreover it is clear that $F(-\infty) = +\infty$, $F(+\infty) = 0$, so that F is a one-to-one map of the reals onto the positive reals. Hence (A2) has a unique solution $\bar{\beta}$; which was assertion (1) of the text.

Now if $\beta = \beta_0$ is constant then from (10), $B(t,s) = (t-s)\beta_0$, whence the integral in (A2) is

$$\int_0^t e^{-\beta_0(t-s)} ds = \frac{1 - e^{-\beta_0 t}}{\beta_0} \ .$$

Then the unique solution of (A2) is obviously $\bar{\beta} = \beta_0$, which was assertion (2).

Given this, the second part of assertion (3) follows from the first. To prove the first simply notice that if β increases pointwise then $\int e^{-B(t,s) ds}$ decreases, so that $\bar{\beta}$ in (A2) must increase because F is monotone decreasing.

Finally let us demonstrate the identity (13). To that end, let us first write out $\langle\, \beta(t) \,\rangle$ in the form (which follows immediately from (11) and (12)),

$$\langle\, \beta \,\rangle \ = \ \frac{\bar{\beta}}{1 - e^{-t\bar{\beta}}} \int_0^t e^{-B(t,s)} \ \beta(s) \ ds \tag{19}$$

Next observe that $\partial/\partial s \ e^{-B(t,s)} = \beta(s)\, e^{-B(t,s)}$, and substitute this into (19) to obtain

$$\langle\, \beta \,\rangle \ = \ \frac{\bar{\beta}}{1 - e^{-t\bar{\beta}}} \int_0^t ds \ \frac{\partial}{\partial s} \ e^{-B(t,s)} \ = \ \frac{\bar{\beta}}{1 - e^{-t\bar{\beta}}} \ \left(1 - e^{-B(t,0)}\right) \ ,$$

from which (13) follows immediately.

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