# An inside view of the tensor product ⋆

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**Abstract**

Given a vector-space $V$ which is the tensor product of vector-spaces $A$ and $B$, we reconstruct $A$ and $B$ from the family of simple tensors $a \otimes b$ within $V$. In an application to quantum mechanics, one would be reconstructing the component subsystems of a composite system from its unentangled pure states. Our constructions can be viewed as instances of the category-theoretic concepts of functor and natural isomorphism, and we use this to bring out the intuition behind these concepts, and also to critique them. Also presented are some suggestions for further work, including a hoped-for application to entanglement entropy in quantum field theory.

*Keywords and phrases*: tensor product, tensor structure, simple vector, subsystem, unentangled, intrinsic definition, square-construction, functor

**Contents**



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This article is dedicated to my friend, A.P. Balachandran, on the occasion of his 85th birthday. With his knack for discerning concrete implications of abstract mathematical relationships, maybe he'll think of an unexpected use in physics for the conception of tensor product proposed herein!

## 1. Introduction

It would hardly be possible to review all the ways in which tensors enter into physics. General Relativity and Quantum Field Theory would not exist without certain individual tensors or tensor-fields, like the Lorentzian metric, the Riemann curvature, or the stress-energy tensor, but it is perhaps in abstract quantum theory where the concept of tensor-product itself, and of the corresponding *product-space* is most prominent. The reason, of course, is that insofar as one deals with state-spaces of quantal "systems", the tensor-product furnishes the construction that combines the state-spaces of two or more subsystems into that of the larger compound system or "whole".

Given this role of tensor-product, it could be unsettling that aside from its dimension, the resulting state-space (call it $V$) appears to remember nothing about the constituent spaces whose product it is. For example, because $12 = 4 \times 3 = 2 \times 6$, a given state-space of dimension 12 that arose by combining spin 3/2 with spin 1 might equally well be describing a composite of spins 1/2 and 5/2. Thus arises the following mathematical question.

Suppose that a certain vector space $V$ is the tensor product of spaces $V_1$ and $V_2$. What extra information do you need in order to recover $V_1$ and $V_2$ from $V$? Or to put the question another way: What does it mean for $V$ to carry the structure of a tensor product space?

To bring this question into sharper focus, imagine that instead of being a tensor-product, $V$ were a direct sum, as it would be for example if $V_1$ described an ionized Hydrogen atom, while $V_2$ described the same atom in unionized form. Then the appropriate state-space would be $V = V_1 \oplus V_2$; and the same equation would describe other mutually exclusive alternatives, like an alpha particle being inside vs. outside a nucleus, or a molecule being ortho-hydrogen vs. para-hydrogen. To our question about tensor products, the counterpart in such cases would be: What does it mean for $V$ to carry the structure of a direct sum? Here however the answer is simple. One only needs to indicate $V_1$ and $V_2$ as *subspaces* of $V$. These two "parts" of $V$ are contained bodily within the whole, and every $v \in V$ is uniquely a sum, $v = v_1 + v_2$. A "direct sum structure" for $V$, in other words, is nothing but a pair of complementary subspaces of $V$.

Why isn't the case of a tensor-product space $V$ equally straightforward? To appreciate what's different, let $V = V_1 \otimes V_2$ be a tensor product space. The first thing one can notice is that $V_1$ and $V_2$ are no longer contained within $V$ in any obvious way. Moreover, the analog of $v = v_1 + v_2$ in a direct sum, would be $v = v_1 \otimes v_2$, but how could one express this within $V$, given that vector-spaces are by definition endowed with a notion of sum but not of product! And lacking the operation $\otimes$ within $V$, how could you recover $V$ from $V_1$ and $V_2$, even if two such subspaces of $V$ could be identified? The development that follows will answer these questions and show that the space $\mathfrak{S}$ of *simple vectors* in $V$ — elements of $V$ of the form $v = v_1 \otimes v_2$ — can play the structural role that the complementary subspaces played in the case of direct sum.

Although our questions are physically inspired, they are purely mathematical, and the answers we will come upon must surely be well known in some circles, even if they haven't shown up in the literature I'm familiar with. Nor do I know whether the constructions we will explore have any deeper physical ramifications. They do, however, offer a more intrinsic way to conceive of the tensor product (an "inside view" as one might say); and a more intrinsic conception, more often than not, illuminates and deepens one's intuition. As anyone who has taught a course in Relativity, Differential Geometry, or "Mathematical Methods" can attest, there's something about the concept of tensor that intuition finds hard to grasp. Perhaps it's no coincidence then that quantum entanglement, the mathematics of which is that of tensor product spaces, also seems counter-intuitive to so many people.

In this situation, the "*square construction*" on which our development will rest, offers a complementary way to think about tensor products, a way that starts not with the individual factor-spaces, $V_1$ and $V_2$, but with the product space $V$ itself. Inasmuch as this more analytical approach shifts the main emphasis onto the simple vectors within $V$, and inasmuch as these simple vectors in a quantum context are precisely the unentangled state-vectors, our development arguably makes better contact with physical intuition than the more formal definitions one usually encounters. At the very least, it offers an alternative to more familiar ways of approaching the topic. After all, the more ways one has to think about a subject, the better the prospect that at least one of them will be able to provide the key to the deeper understanding that one is seeking.

The main ingredients of our constructions are presented in Sections 2-5 below. Sections 2 and 3 are preparatory, while the constructions themselves appear in Sections 4 and 5.

In Section 6 we ask whether the results of Sections 4 and 5 can be regarded as fully capturing the structure of $V$ as a tensor product; and we show in some detail how to make this type of question precise in terms of the category-theoretic concepts of functor and natural transformation, to which we provide a brief introduction. In this connection, we also point out a certain shortcoming of the functor concept itself.

In Section 7 we suggest some extensions of our constructions to tensor products of three or more spaces, or to products of a single space with itself, in which case symmetry-conditions come into play (bosonic, fermionic, or nonabelian). We also speculate that a suitable generalization of the notion of simple vector to infinite dimensions could help to clear away the mathematical obstructions (involving type-III von Neumann algebras, where tensor products as normally defined are not available) that prevent one from understanding entanglement-entropy (with its need for a cutoff) in terms of reduced density matrices.

In what follows, we will assume, unless otherwise specified, that all vector spaces are real and finite dimensional. Nothing would change if we replaced the field $\mathbb{R}$ with the field $\mathbb{C}$, but it's convenient for exposition to pick one or the other and stick with it. We will also assume without special mention that the spaces $V_1$ and $V_2$ are distinct from each other.

## 2. Posing the problem

Our question asks for a more intrinsic definition of tensor product, or as we worded it above: What could play the role of a tensor-product structure for $V$? We will contemplate three possible answers to this question, and in doing so we will always assume that the two spaces of which $V$ is a product are distinct from each other. Often this will not matter, but sometimes it would make a difference, notably in definition (1) of the paragraph after next, and then farther below in the "Second answer" to our main question.

Before suggesting answers to our main question, however, it seems advisable to dwell for a moment on the more common definitions of tensor product. Different authors favor different ones, and the answers to our questions will tend to take on different forms, depending on which definition one has in view. What, then, are some of the popular definitions [1] of the space $V_1 \otimes V_2$ and of the tensors therein?

(1) an element of $V_1 \otimes V_2$ is a numerical matrix whose entries depend on a choice of bases for $V_1$ and $V_2$ and transform in a certain way when these bases are changed. (This might be the oldest definition.)

(2) an element of $V_1 \otimes V_2$ is a linear mapping between two vector spaces, for example a linear mapping from $V_2^*$ to $V_1$, where $V_2^*$ is the dual space of $V_2$

(3) an element of $V_1 \otimes V_2$ is an equivalence class of formal sums of symbols, $\alpha \otimes \beta$, where $\alpha \in V_1$ and $\beta \in V_2$

(4) "the" space $V_1 \otimes V_2$ is any solution of a certain "universal mapping problem" involving bilinear functions from $V_1 \times V_2$ to an arbitrary vector space $Y$. (Thereby a bilinear function from $V_1 \times V_2$ to $Y$ induces a unique linear mapping from $V_1 \otimes V_2$ to $Y$.)

Remark Notice that definition (1) refers to independently chosen bases for $V_1$ and $V_2$. Were we to assume that $V_1$ and $V_2$ were literally the same space, only a single basis would come into play. Tensors in this vein are common in GR and differential geometry, with $V_1 = V_2$ being the tangent space to a point of spacetime. In quantum mechanics on the other hand, distinct factor-spaces are typical, albeit not in the case of indistinguishable particles.

In finite dimensions, all these definitions are provably equivalent. In infinite dimensions one must distinguish between the so-called algebraic tensor product and various topological tensor products, not all of which are the same in general. For our purposes, entering into those subtleties would be too much of a distraction. [2] Instead, we will work throughout in finite dimensions. In the jargon of category theory, each of these definitions yields a "functor" from pairs of vector spaces to vector spaces, and the statement that (in finite dimensions) they are all equivalent asserts that between any two of the functors there is an invertible "natural transformation". Now back to our main question and some possible answers to it.

*First possible answer*. The most direct and obvious answer, but at the same time the least informative, is that a tensor-product structure for $V$ is an isomorphism between $V$ and a space of the form $V_1 \otimes V_2$. This is a good start, but it has the drawback that the auxiliary spaces $V_1$ and $V_2$ are not derived from $V$, with the consequence that different choices of them would strictly speaking define different tensor-product structures for $V$. We could address this difficulty by forming equivalence-classes under isomorphisms of $V_1$ and $V_2$, but let's instead continue on to the second and third proposals.

*Second answer*. The second possible answer to our main question, already more concrete and "intrinsic", hearks back to definition (1) in the above list. In order to represent an element of $V = V_1 \otimes V_2$ as a numerical matrix, one needs a basis of $V$ whose members are themselves organized into a matrix. Specifically, if a list of vectors $e_j \in V_1$ furnishes a basis for $V_1$ and a second list of vectors $f_k \in V_2$ furnishes a basis for $V_2$, then the products $e_j \otimes f_k$ furnish a basis for $V_1 \otimes V_2$ whose members array themselves in a rectangular matrix with rows labeled by $j$ and columns by $k$. In such a basis the matrix representing a simple vector $v_1 \otimes v_2$ will be a matrix-product of the form, column-vector $\times$ row-vector. (Such a "special basis" is precisely an isomorphism between $V$ and $\mathbb{R}^m \otimes \mathbb{R}^n$, where $m$ and $n$ are the respective dimensions of $V_1$ and $V_2$.)

We could thus answer that a tensor-product structure for $V$ is a basis for $V$ organized into a rectangular matrix. Unfortunately this won't quite do, because many other bases will define the same product-structure. First of all, one might swap rows with columns, which amounts to writing $V_2 \otimes V_1$ instead of $V_1 \otimes V_2$. This does nothing. However one can also replace each of the two bases by some other basis for the same space, which doesn't change the spaces $V_1$ or $V_2$ themselves but only their representations. We are thus led to identify a tensor-product structure for $V$ as an equivalence-class of bases, parameterized by $G = GL(n_1) \times GL(n_2)$ where $n_i = \dim(V_i)$. But even this is not quite correct since it is always possible to rescale the basis for $V_1$ by some factor, while rescaling the basis for $V_2$ in the opposite way. Since this doesn't affect the resulting basis for $V_1 \otimes V_2$, we conclude that $G$ is really the quotient group $GL(n_1) \times GL(n_2) \, / \, GL(1)$. Here of course $GL(n)$ is the group of invertible $n \times n$ matrices.

The fact that true group is $GL(n_1) \times GL(n_2) \, / \, GL(1)$ and not simply $GL(n_1) \times GL(n_2)$ seems a detail, but it is actually telling us something that will show up again in our deliberations below. From a tensor-product structure for $V$, we cannot fully reconstruct the factor-spaces $V_1$ and $V_2$; we can obtain them only up to a joint scaling ambiguity.

REMARK In a classical (non-quantum) context, a composite system would be described by a cartesian product, $A \times B$. In that case, the counterpart of a rectangular basis would just be (for discrete spaces $A$ and $B$) a rectangular list of elements of $A \times B$, and the story would more or less end there. * Tensor products are more subtle than cartesian products, however, and there's a third possible answer to our question which is still more concrete and intrinsic than an equivalence class of bases.

*Third answer*. The third possible answer to our question, and the one which the rest of this paper will explore, is that the tensor-product structure for $V = V_1 \otimes V_2$ can be taken to be the subspace $\mathfrak{S}$ of *simple vectors*:

$$\mathfrak{S} = \{\alpha \otimes \beta \,|\, \alpha \in V_1, \beta \in V_2\} \tag{1}$$

As we will see, there exist explicit constructions that take you from $\mathfrak{S} \subseteq V$ to (copies of) $V_1$ and $V_2$, and thence back to $V$.

REMARK In quantum language, $\mathfrak{S}$ would be the set of unentangled state-vectors. Obviously there's something special about them, but mathematically they are only the first in a hierarchy of successively more generic tensors, those of ranks 2, 3, etc, where the "rank" of $v$ is the minimum number of simple vectors of which it is a sum (quantum mechanically the number of terms in a Schmidt decomposition of $v$.) †

## 3. The space $\mathfrak{S}$ of simple vectors in $V$

Since it is related quadratically to $V_1$ and $V_2$, the set $\mathfrak{S}$ of simple vectors obviously will not be a linear subspace of $V$ in general, but it will be foliated by two families of linear subspaces, which we will denote by $\mathcal{M}_1$ and $\mathcal{M}_2$.

Before demonstrating this, let us deal with two trivial cases that don't fit easily into the general pattern, In the most trivial case, both $V_1$ and $V_2$ are one-dimensional: $\dim(V_1) = \dim(V_2) = 1$. Both are then copies of $\mathbb{R}$, as also is $V = V_1 \otimes V_2$. In this case every $v \in V$ is plainly a simple vector, and so $\mathfrak{S}$ is all of $V$. Conversely, given that $\dim(V) = 1$, and since we know in general that $\dim V_1 \otimes V_2 = \dim V_1 \times \dim V_2$, we know immediately that both $V_1$ and $V_2$ are isomorphic to $V$ itself. In a reconstruction of $V_1$ and $V_2$, we can thus do no better than to take both to be copies of $V$, and this suffices. The only small subtlety shows up when, having passed from $V$ to $V_1$ and $V_2$, we seek to

* The counterpart of $G$ would be the product of the permutation groups of $A$ and $B$

† Algebraic Geometry has given the name "Segre variety", not quite to $\mathfrak{S}$ itself, but to the set of rays in $\mathfrak{S}$

reconstruct $V$ as $V_1 \otimes V_2 = V \otimes V$, but have to face the fact that although $V$ is isomorphic to $V \otimes V$, the isomorphism is not canonical.$^\flat$

In the second trivial case, $\dim V_1 > 1$ while $\dim V_2 = 1$ (or vice versa). Here again $\mathfrak{S}$ is trivially all of $V$. (By definition any $v \in V$ is a sum of terms of the form $a \otimes b$ for $a \in V_1$ and $b \in V_2$, but since all nonzero $b$ are proportional to each other, all the $b$ can be taken equal, whence $v = a_1 \otimes b + a_2 \otimes b + \cdots = (a_1 + a_2 \cdots) \otimes b \in \mathfrak{S}$.) It follows that $V = V_1 \otimes V_2 \simeq V_1 \otimes \mathbb{R} = V_1$ (where '$\simeq$' signifies isomorphic-to). Conversely, whenever $\mathfrak{S} = V$, we can construct spaces $V_1$ and $V_2$ by taking $V_1$ to be $V$ and $V_2$ to be any one-dimensional vector space, for example the subspace of $V$ given by $\mathbb{R}\, b_0$, where $b_0$ is any fixed vector$^\star$ in $V$. The now-familiar scaling-ambiguity corresponds then to the undetermined normalization of $b_0$.

Notice in these two examples that $V_1$ was identified with a maximal linear subspace of $\mathfrak{S}$. Although completely trivial in the two examples, this observation will be the basis of our reconstruction of $V_1$ and $V_2$ in the generic case. In seeking to understand the linear subspaces of $\mathfrak{S}$, we will need a few "obvious" facts about tensor products which we will now review in the spirit of definition (3) mentioned in the previous section.

Recall then that any tensor $T \in V_1 \otimes V_2$ is a sum of simple tensors, i.e. a sum of products of vectors from $V_1$ with vectors from $V_2$:

$$T = \sum_j a_j \otimes b_j \tag{2}$$

To fully characterize the space $V_1 \otimes V_2$, however, one needs to specify which such sums are equal to which others, or equivalently which expressions $T$ equal the zero tensor. Intuitively the answer is that $T = 0$ iff it is forced to vanish by the combining rules for the symbols $a \otimes b$ together with the linear dependences among the vectors of $V_1$ and $V_2$. This criterion is implicit in the aforementioned definition (4), but it is more useful to express it algorithmically.

Rule Provided that the vectors $a_j$ in (2) are linearly independent, $T = 0$ if and only if all of the $b_j$ vanish.

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$\flat$ In some sense this is just "dimensional analysis". If the elements of $V$ were "lengths", then those of $V \otimes V$ would be squared lengths.

$\star$ The reason for this particular choice will become clear soon. Notice also that we could of course have exchanged the roles of $V_1$ and $V_2$.

(Obviously the same rule will hold true if we exchange the roles of $a_j$ and $b_j$.) As stated, the rule wants the $a_j$ to be linearly independent. If they are not, then some of them can be expressed as linear combinations of the others, and one should do this before applying the rule. Thus, an *algorithm* for deciding whether $T = 0$ consists in first writing any redundant $a_j$ in terms of the others, second expanding out the resulting expression to put $T$ into the form (2), and third applying the rule as stated.

As a trivial consequence of this rule, we learn that $a \otimes b$ is nonzero if both $a$ and $b$ are. In stating the following further consequences, we will interpret $a \propto b$ to mean that either $a = \lambda b$ or $b = \lambda a$, $\lambda \in \mathbb{R}$.

Lemma 1. *If $\alpha$ and $\beta$ are nonzero then $\alpha \otimes \beta = \alpha' \otimes \beta' \Rightarrow \alpha' \propto \alpha$ and $\beta' \propto \beta$.*

Proof Were $\alpha'$ not proportional to $\alpha$, they would be linearly independent. Our "Rule" would then imply that $\alpha \otimes \beta - \alpha' \otimes \beta'$ could not vanish. Therefore $\alpha' \propto \alpha$, and by symmetry $\beta' \propto \beta$.

Lemma 2. *Let $\alpha \otimes \beta \in \mathfrak{S}$ and $\alpha' \otimes \beta' \in \mathfrak{S}$ be nonzero simple vectors. If their sum is also simple then either $\alpha' \propto \alpha$ or $\beta' \propto \beta$.*

Proof (by contradiction). Assume that $\alpha' \not\propto \alpha$ and $\beta' \not\propto \beta$. The four terms $\alpha \otimes \beta$, $\alpha' \otimes \beta'$, $\alpha \otimes \beta'$, $\alpha' \otimes \beta$ are then (by a simple application of the Rule) linearly independent. By hypothesis $\alpha \otimes \beta + \alpha' \otimes \beta' = \gamma \otimes \delta$ for some $\gamma$ and $\delta$. Appealing once again to the Rule, and remembering that $\alpha'$ is independent of $\alpha$, we conclude that $\gamma$ must be a linear combination of $\alpha$ and $\alpha'$; similarly $\delta$ must be a linear combination of $\beta$ and $\beta'$. But then $\gamma \otimes \delta$, when expanded out, could not contain the required terms, $\alpha \otimes \beta$ and $\alpha' \otimes \beta'$ without also containing terms in $\alpha \otimes \beta'$ and $\alpha' \otimes \beta$.

Returning now to the analysis of $\mathfrak{S}$, and recalling that we have already disposed of the possibility that either $\dim V_1 = 1$ or $\dim V_2 = 1$, we can assume for now that $\dim V_1 \geq 2$ and $\dim V_2 \geq 2$, this being where the typical structure of $\mathfrak{S}$ reveals itself, namely that of the two foliations $\mathcal{M}_1$ and $\mathcal{M}_2$ already alluded to but not yet defined. For the time being, we will define $\mathcal{M}_1$ and $\mathcal{M}_2$ as follows. Soon, we will define them intrinsically (meaning directly from $V$ and $\mathfrak{S}$ alone), whereupon (3) and (4) will shed their status as definitions and become theorems. The members of $\mathcal{M}_1$ will be the subsets of $\mathfrak{S}$ of the form $V_1 \otimes \beta$ for $\beta \in V_2$, and likewise for $\mathcal{M}_2$:

$$\mathcal{M}_1 = \{V_1 \otimes \beta \mid \beta \in V_2\} \tag{3}$$

$$\mathcal{M}_2 = \{\alpha \otimes V_2 \mid \alpha \in V_1\} \tag{4}$$

(Here of course our notation means that, e.g, $V_1 \otimes \beta = \{\alpha \otimes \beta \,|\, \alpha \in V_1\}$.)

We want to prove first, that every $M$ in either $\mathcal{M}_1$ or $\mathcal{M}_2$ is a *maximal linear subspace of* $\mathfrak{S}$; second that $\mathcal{M}_1$ and $\mathcal{M}_2$ exhaust the maximal linear subspaces of $\mathfrak{S}$; third that $M, N \in \mathcal{M}_1$ and $M \neq N \Rightarrow M \cap N = \{0\}$ (and likewise for $\mathcal{M}_2$); and fourth that $M \in \mathcal{M}_1, N \in \mathcal{M}_2 \Rightarrow \dim(M \cap N) = 1$.

Why are the members of $\mathcal{M}_1$ (for example) maximal linear subspaces of $\mathfrak{S}$? That $M = V_1 \otimes \beta$ is a linear subspace is obvious, but why is it maximal? Well, any simple vector not in $M$ must take the form $\alpha' \otimes \beta'$ with $\beta'$ independent of $\beta$. Choose also an $\alpha \in V_1$ that is independent of $\alpha'$ (which is always possible since $\dim V_1 > 1$), and notice that $\alpha \otimes \beta \in M$. If we could adjoin $\alpha' \otimes \beta'$ to $M$ then $\alpha' \otimes \beta' + \alpha \otimes \beta$ would also have to be in $M$, and therefore simple, contrary to Lemma 2 above. †

And why does every maximal linear subspace of $\mathfrak{S}$ have to belong to either $\mathcal{M}_1$ or $\mathcal{M}_2$? Well, let $M$ be such a subspace, and let $\alpha \otimes \beta \in M$. Certainly $\alpha \otimes \beta$ alone is not maximal (it belongs to $V_1 \otimes \beta$, for example), so let $\alpha' \otimes \beta'$ be an independent member of $M$. By the same lemma either $\alpha$ and $\alpha'$ are proportional or $\beta$ and $\beta'$ are proportional, say the latter. Then as we just saw, every other member of $M$ must also take the form $\gamma \otimes \beta$ for some $\gamma \in V_1$, in other words $M \subseteq V_1 \otimes \beta \in \mathcal{M}_1$, whence $M = V_1 \otimes \beta$ since $M$ is maximal.

Third, if $M, N \in \mathcal{M}_1$ are unequal then $M = V_1 \otimes \beta$ and $N = V_1 \otimes \beta'$ with $\beta$ independent of $\beta'$. Hence any $v \in M \cap N$ must satisfy $v = \alpha \otimes \beta = \alpha' \otimes \beta'$ for some $\alpha$ and $\alpha'$. But by Lemma 1, this is impossible unless $v = 0$.

Fourth, if $M \in \mathcal{M}_1, N \in \mathcal{M}_2$ then $M = V_1 \otimes \beta$, $N = \alpha \otimes V_2$ for some $\alpha \in V_1, \beta \in V_2$. If $v \in M \cap N$ then by definition, $v = \alpha' \otimes \beta = \alpha \otimes \beta'$ for some $\alpha' \in V_1, \beta' \in V_2$. The lemma just cited then informs us that $\alpha' \propto \alpha$ and $\beta' \propto \beta$, whence $v = \alpha' \otimes \beta \propto \alpha \otimes \beta$. In other words $M \cap N$ is the 1-dimensional subspace, $\mathbb{R}\, \alpha \otimes \beta$

The essential feature we have discovered is that *any two members of different foliations meet in a ray (a one-dimensional subspace of $V$) and any two distinct members of the same foliation are disjoint.* This lets us determine the foliations $\mathcal{M}_1$ and $\mathcal{M}_2$ simply from a knowledge of $\mathfrak{S} \subseteq V$, without any further recourse to how $V$ arose as a tensor product: if $M$ and $N$ are elements of the set $\mathcal{M}$ of all maximal linear subspaces of $\mathfrak{S}$, then they belong to the same foliation if and only if they are disjoint, and this criterion is guaranteed to

† What we are effectively proving could be reduced to a lemma to the effect that every linear subspace of $\mathfrak{S}$ has the form $W \otimes \beta$ or $\alpha \otimes W$, for some vector-subspace $W$ of $V_1$ or $V_2$

produce exactly two disjoint subsets of $\mathcal{M}$, which we can label as $\mathcal{M}_1$ and $\mathcal{M}_2$. Henceforth, we will adopt this *intrinsic definition* of $\mathcal{M}_1$ and $\mathcal{M}_2$, which we can record in the following two maps that associate with each simple vector in $V$ the two maximal linear subspaces of $\mathfrak{S}$ to which it belongs.

Definition Let $v \in \mathfrak{S}$. Then $\pi_1(v)$ [resp. $\pi_2(v)$] is the unique maximal linear subspace of type $\mathcal{M}_1$ [resp. $\mathcal{M}_2$] that contains $v$.

Equations, (3)-(4), are hereby no longer definitions but theorems which apply whenever we can exhibit vector-spaces $V_1$ and $V_2$ such that $V = V_1 \otimes V_2$.

With these observations, we have taken a first step in recovering the tensor product structure of $V$ from $\mathfrak{S}$. In fact, one sees from (3) and (4) that each $M_1$ in $\mathcal{M}_1$ is a copy of $V_1$ and each $M_2$ in $\mathcal{M}_2$ is a copy of $V_2$. In the following section, we will build on our knowledge of $\mathcal{M}_1$ and $\mathcal{M}_2$ to recover fully the ray-spaces associated to $V_1$ and $V_2$, and then to recover $V_1$ and $V_2$ themselves up to scale.

## 4. How to recover $V_1$ and $V_2$ up to scale

Our ultimate aim is to find a construction that, relying on nothing more than the set $\mathfrak{S}$ of simple vectors in $V$, will resolve the latter into its two factors (as uniquely as possible), and then to discover how to rebuild $V$ as the tensor product of these factors. This will take place in Section 5, and not everything from the present section will be needed there. If you are reading these lines, you might thus want to skip over the present section in order to appreciate the great simplicity of the final constructions. On the other hand, the present section, as well as providing much of the background for Section 5, will also show how, in becoming aware of the two spaces $\mathcal{M}_1$ and $\mathcal{M}_2$, we have *already* recovered from $\mathfrak{S}$ the *rays* of $V_1$ and $V_2$, which in a quantum context means we have already recovered, if not the respective subsystems themselves, then at least their "pure states".

To appreciate this fact, recall that when $V = V_1 \otimes V_2$, any member $M$ of $\mathcal{M}_2$ can be expressed in the form (4). But the subspace $M = \alpha \otimes V_2$ determines and is determined by the ray, $\mathbb{R}\,\alpha \subseteq V_1$. The points of $\mathcal{M}_2$ are thus in bijective correspondence with the rays of $V_1$, and likewise for $\mathcal{M}_1$ and $V_2$. Introducing the notation $\mathbf{P}V$ for the projective space formed from the rays of any vector-space $V$, we can therefore assert that

$$\mathbf{P}V_1 = \mathcal{M}_2 \qquad \text{and} \qquad \mathbf{P}V_2 = \mathcal{M}_1 \tag{5}$$

Of course, there's more to it than this, because so far, we have only introduced $\mathcal{M}_1$ and $\mathcal{M}_2$ as sets without further structure. In order to fully corroborate the claim that $\mathbf{P}V_1 = \mathcal{M}_2$, we need to present $\mathcal{M}_2$ as the set of rays of some intrinsically defined vector space, this being one way to equip it with a projective structure. In the course of doing so, we will also see how to get our hands on $V_1$ itself up to scale.

Let's first see the procedure per se and then return to see more fully why it works. To get started, select arbitrarily any $M \in \mathcal{M}_1$ and let $P$ be the restriction of $\pi_2$ to $M$. It is not hard to see that $P : M \to \mathcal{M}_2$ sets up a one-to-one correspondence between the rays in $M$ and the points of $\mathcal{M}_2$. By definition, if $v \in M$ then $P(v) = \pi_2(v)$ is the unique maximal linear subspace in $\mathcal{M}_2$ that contains $v$; being linear, it also contains the entire ray, $\mathbb{R}\, v$. Furthermore, $P$ is trivially surjective because for any $N \in \mathcal{M}_2$, $M \cap N$ is (as observed earlier) a ray $\ell$ in $M$ that gets mapped by $P$ to $N$ itself. This also proves that $P$ is injective (on the *rays* of $M$) because any other ray in $M$ that was mapped to $N$ by $P$ would by definition have to lie in $M \cap N$ and therefore coincide with $\ell$.

The mapping, $P : M \to \mathcal{M}_2$, is what we were looking for, but it remains to demonstrate that any other $M' \in \mathcal{M}_1$ would have induced the same projective structure on $\mathcal{M}_2$. For this, it suffices to find a linear isomorphism between $M$ and $M'$ that commutes with the corresponding projections. In other words, with $P'$ taken to be the restriction of $\pi_2$ to $M'$, we should seek an isomorphism $f : M \to M'$ such that $P = P' \circ f$. Such an $f$ would induce an isomorphism $\hat{f} : \mathbf{P}M \to \mathbf{P}M'$, and it is actually easier to characterize this isomorphism intrinsically than to exhibit $f$ itself. Let us therefore define $\hat{f}$ first, and only then consider how to lift it to a linear map $f$. It turns out that the Ansatz,

$$\hat{f}(M \cap N) = M' \cap N \tag{6}$$

(where $N$ is an arbitrary element of $\mathcal{M}_2$) does what is needed. In particular, if $M''$ is a third element of $\mathcal{M}_1$, then the isomorphisms $M \to M' \to M''$ defined by (6) obviously compose consistently.

Our remaining task is to lift the just-constructed mapping, $\hat{f} : \mathbf{P}M \to \mathbf{P}M'$, to a linear function, $f : M \to M'$. Given that for $v \in M$, the mapping $\hat{f}$ already determines the ray in $M'$ to which $v$ should go, the only further input needed to define $f(v)$ is its normalization. Although this seems a tiny bit of extra information, the construction via which we will obtain it is surprisingly intricate. In fact, it is not really needed for present purposes; all we really need to know is that a linear lift $f$ exists, which could be proven more easily. If nevertheless we take the trouble to construct $f$ explicitly, it is because doing so will introduce us to a certain type of "simple-square" that will play an important role in the next section.

Fix spaces $M, M' \in \mathcal{M}_1$ as above, and let $\ell_0$ be any ray in $M$, with $\ell_0' = \hat{f}(\ell_0)$ being the corresponding ray in $M'$, as given by (6). We know that $f$ will take any point in $\ell_0$ to some point in $\ell_0'$. Given now some arbitrarily chosen reference vector, $v_0 \in \ell_0$, we need to decide which vector in $\ell_0'$ will be $f(v_0)$, and it turns out that this decision determines $f$ fully. Let $v_0'$ be the vector selected to be $f(v_0)$. The problem then is to determine $f(v)$ when $v$ belongs to some other ray $\ell \subseteq M$. That is, we need to figure out where $f(v)$ lies along the ray $\ell' = \hat{f}(\ell)$.

This problem admits a generic case and a couple of special cases. In the generic case, $v_0$, $v_0'$, and $v$ are all linearly independent. Consider then an arbitrary $v' \in \ell'$ and the *square*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} v_0 & v_0' \\ v & v' \end{pmatrix} \tag{7}$$

whose elements belong to the rays

$$\begin{pmatrix} \ell_0 & \ell_0' \\ \ell & \ell' \end{pmatrix} . \tag{8}$$

By construction (cf. (6)),

$$\pi_1 a = \pi_1 c\,, \quad \pi_1 b = \pi_1 d\,, \quad \pi_2 a = \pi_2 b\,, \quad \pi_2 c = \pi_2 d\,. \tag{9}$$

Consequently, the two row-sums and the two column-sums belong to $\mathfrak{S}$ (i.e. all four sums are simple vectors in $V$), but what about the overall sum, $a+b+c+d$? In the answer to this question lies the key to our construction of $f$. In fact (as we will prove shortly) this sum meets $\mathfrak{S}$ for precisely one point $v'$ in the ray $\ell'$, and by setting $f(v) = v'$ we define $f$ uniquely on the ray $\ell$. Doing the same for the other rays in $M$, we will obtain a function $f : M \to M'$ which is linear, unique up to a multiplicative prefactor, and whose action on rays is by definition that of $\hat{f}$.

So much for the generic case. Before turning to the special cases, observe that just from (9) alone, we can write the rays in (8) as

$$\begin{pmatrix} M \cap N & M' \cap N \\ M \cap N' & M' \cap N' \end{pmatrix} , \tag{10}$$

where $M = \pi_1 a = \pi_1 c$, $N = \pi_2 a = \pi_2 b$, $M' = \pi_1 b = \pi_1 d$, $N' = \pi_2 c = \pi_2 d$. The generic case just treated corresponded to an array (10) in which the subspaces, $M, M', N, N'$, were all distinct, and correspondingly the vectors, $a, b, c, d$, in (7) were linearly independent. The special cases we still need to treat are those in which $M = M'$ or $N = N'$.

Consider first the special case where $N = N'$, or equivalently, $\ell = \ell_0$. Here we know the answer trivially because $v = \lambda v_0$ for some scalar $\lambda$, whence $v' = f(v) = f(\lambda v_0) = \lambda f(v_0) = \lambda v_0'$. The square in (7) thus assumes the form,

$$\begin{pmatrix} a & b \\ \lambda a & \lambda b \end{pmatrix} \tag{11}$$

The other special case is that where $M = M'$, or equivalently (since, as we know, any two elements of $\mathcal{M}_1$ are either equal or disjoint), $\ell_0' = \ell_0$. Here $f$ is just mapping $M$ to itself, an obvious solution for which would be to take $f$ to be the identity map. However, we could equally well take it to be a multiple of the identity by a scalar $\mu$, in which case our square would take on the appearance,

$$\begin{pmatrix} a & \mu a \\ c & \mu c \end{pmatrix}, \tag{12}$$

a form that follows immediately from (11) by symmetry. For completeness, let us also record the doubly special case where $M = M'$ and $N = N'$ both hold, leading to a square of the design,

$$\begin{pmatrix} a & \mu a \\ \lambda a & \lambda \mu a \end{pmatrix}, \tag{13}$$

as one sees by combining (11) with (12). All these special cases, (11) – (13), can be obtained from the generic case by forming limits. Amalgamating these special cases with the generic one, we arrive at the following definition.

Definition A *square* (or *simple square*) is a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of simple vectors which satisfy (9), and which in the generic case satisfy $a + b + c + d \in \mathfrak{S}$, or in the special cases take on one of the forms (11)–(13).

The reason for separating the generic from special cases in the definition is that $a+b+c+d \in \mathfrak{S}$ suffices in the generic case, but not in the special cases. Of course, it holds in the latter cases too, albeit it is trivial there. It's also worth noting that given any simple square, one can multiply any row or column by a scalar without invalidating its status as a square. And for completeness, let us recall from above that the two row-sums and the two column-sums also belong to $\mathfrak{S}$.

This completes the description of our procedure for defining $f$. In order to understand why it works, let's "look behind the curtain" to see what our squares amount to when expressed in terms of vectors in $V_1 \otimes V_2$. (This should also help to illuminate the rather abstract development we have been following in this section.) Recall that the four rays in (8) can also be written as the intersecting subspaces exhibited in (10). Now by equations (3) and (4), $M = V_1 \otimes \beta_0$ for some $\beta_0 \in V_2$, while $N = \alpha_0 \otimes V_2$ for some $\alpha_0 \in V_1$, and similarly $M' = V_1 \otimes \beta$, $N' = \alpha \otimes V_2$, for some $\alpha$ and $\beta$. Without loss of generality we can therefore write (7) in the form,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha_0 \otimes \beta_0 & \alpha_0 \otimes \beta \\ \alpha \otimes \beta_0 & \lambda \, \alpha \otimes \beta \end{pmatrix} \tag{14}$$

where $\lambda$ is some unknown coefficient of proportionality. This form makes it plain that the row- and column-sums are indeed simple, for example $a + b = \alpha_0 \otimes (\beta_0 + \beta)$. As for the overall sum, $a+b+c+d$, it will be the simple vector, $(\alpha_0+\alpha) \otimes (\beta_0+\beta)$ *provided that* $\lambda = 1$. Were $\lambda \neq 1$ on the other hand, the same sum would equal $(\alpha_0+\alpha) \otimes (\beta_0+\beta) + (\lambda-1)\alpha \otimes \beta$, which according to Lemma 2, could be simple only if $\alpha$ were proportional to $\alpha_0$ or $\beta$ were proportional to $\beta_0$, meaning we'd be back in one of the special cases we disposed of earlier.

In summary, consider a square of simple vectors belonging to rays of the form exhibited in (10) with $M, M' \in \mathcal{M}_1$ and $N, N' \in \mathcal{M}_2$, and assume we are in the generic case where $M \neq M'$, $N \neq N'$. On condition that the sum of all four simple vectors is itself simple, any three of them determine the fourth uniquely. The vectors must in that case "secretly" take the form (14) with $\lambda = 1$:

$$\begin{pmatrix} \alpha_0 \otimes \beta_0 & \alpha_0 \otimes \beta \\ \alpha \otimes \beta_0 & \alpha \otimes \beta \end{pmatrix} \tag{15}$$

Our special cases correspond to $\alpha_0 \propto \alpha$ and/or $\beta_0 \propto \beta$, and they also fit the pattern (15), which accordingly represents the universal form that a square assumes when one views it "from behind the curtain".

Returning to the task of lifting $\hat{f} : \mathbf{P}M \to \mathbf{P}M'$ to a linear isomorphism, $f : M \to M'$, we can now see that the construction of $f(v)$ following equation (9) does indeed do the job, because it maps $M = V_1 \otimes \beta_0$ to $M' = V_1 \otimes \beta$ by carrying $\alpha \otimes \beta_0 \in M$ to $\alpha \otimes \beta \in M'$, a correspondence which is plainly linear when $\alpha$ varies. Of course, the fact that $f$ is a lift of $\hat{f}$ cannot determine it uniquely, because any multiple of a lift is another lift. It's thus no accident that our construction involved a free choice of reference vectors, $v_0$ and $v_0'$. A different choice, however, could only alter $f$ by an overall factor, as follows from the general fact that any two linear isomorphisms that induce the same mapping on rays must agree up to scale.$^\flat$ For the same reason, we don't need to check our isomorphisms $f$ for coherence. Given that they cohere on $\mathbf{P}M \to \mathbf{P}M' \to \mathbf{P}M''$, as we already know they do, they must also cohere up to scale on $M \to M' \to M''$, and that's the best we can do.

Taking as input solely the set $\mathfrak{S}$ of simple vectors in $V$, we have now identified with one another the members of $\mathcal{M}_1$ via isomorphisms which are unique up to scale. On one hand we used this to derive from $V$-cum-$\mathfrak{S}$ a canonically given projective space that is naturally isomorphic to $\mathbf{P}V_1$ (the space of "pure states of system-1" in a quantal interpretation). On the other hand, these same identifications produce a vector space that is naturally isomorphic to $V_1$ itself, albeit only modulo a scaling ambiguity. The same procedure applied to $\mathcal{M}_2$ rather than $\mathcal{M}_1$ would obviously recover $\mathbf{P}V_2$ and $V_2$ in the same sense. Our task now is to complete the story by re-building $V$ as the tensor product of the two vector spaces just constructed.

## 5. The analysis and synthesis of a tensor product

Our previous work has already led us to pay close attention to the maximal linear subspaces of $\mathfrak{S} \subseteq V = V_1 \otimes V_2$. Let us now select two such spaces, $W_1 \in \mathcal{M}_1$ and $W_2 \in \mathcal{M}_2$, and then select further a vector $w_0 \in W_1 \cap W_2$ to serve as their common "base point".$^\star$ We have then

$$W_1 = \pi_1 w_0 \ , \qquad W_2 = \pi_2 w_0 \ .$$

$^\flat$ Proof. Call the maps $f$ and $g$ and let $x$ and $y$ be any two independent vectors in their domain with $z = x + y$. By assumption $g(x) = \lambda f(x)$ and $g(y) = \mu f(y)$, and we want to prove that $\mu = \lambda$. By rescaling either $f$ or $g$ if necessary, we can assume that $\lambda = 1$. But then $f(z) = f(x) + f(y)$ would lie in a different ray from $g(z) = f(x) + \mu f(y)$ unless $\mu = 1$ as well.

$^\star$ In Sections 3 and 4 we usually used the letters $M$ and $N$ to denote maximal linear subspaces of $\mathfrak{S}$. The notation, $W_1$, $W_2$, here is chosen to emphasize the parallelism with $V_1$, $V_2$.

We want to demonstrate that $V$ can be construed as the tensor-product of these two spaces.

To that end, and basing ourselves on the concept of "square" introduced in Section 4, we will introduce a new bilinear product, $\overline{\otimes} : W_1 \times W_2 \to V$, as follows. For $w_i \in W_i$ $(i = 1, 2)$, let us define $w_1 \overline{\otimes} w_2$ to be the solution of the following square:

$$\begin{pmatrix} w_0 & w_2 \\ & \\ w_1 & w_1 \overline{\otimes}\, w_2 \end{pmatrix} \tag{16}$$

In other words, $w = w_1 \overline{\otimes}\, w_2$ must satisfy the conditions,

$$w \in \pi_2 w_1 \,\cap\, \pi_1 w_2$$

$$w_0 + w_1 + w_2 + w \,\in\, \mathfrak{S} \ .$$

As we have seen, these conditions determine $w_1 \overline{\otimes}\, w_2$ uniquely in the generic case where $w_1 \notin W_2$ and $w_2 \notin W_1$. In the special cases where this is not true, a scaling ambiguity remains. To supplement (16) for such cases, we can stipulate that $w_0 \overline{\otimes} w_0 = w_0$, and more generally that $w_0 \overline{\otimes} w_2 = w_2$ and $w_1 \overline{\otimes} w_0 = w_1$. These rules † render $w_1 \overline{\otimes}\, w_2$ unique. For example if $w_1 \in W_2$ then $w_1 \in W_1 \cap W_2$, whence $w_1 = \lambda w_0$ since, as always, $\dim(W_1 \cap W_2) = 1$. Therefore $w_1 \overline{\otimes}\, w_2 = (\lambda w_0) \overline{\otimes}\, w_2 = \lambda w_2$ , exactly as in (11).

We learned in the previous section [following eq. (15)] that $\overline{\otimes}$ would be bilinear when defined in this manner. ♭ Therefore (compare definition (2) of tensor-product in Section 2) it induces a linear map $\Phi : W_1 \otimes W_2 \to V$. In fact $\Phi$ is an isomorphism. To see this, let's go back to the representation of $V$ as $V_1 \otimes V_2$, and write $w_0 = \alpha_0 \otimes \beta_0$, $w_1 = \alpha \otimes \beta_0$, $w_2 = \alpha_0 \otimes \beta$. Then as one sees by comparing (15) with (16), $w_1 \overline{\otimes} w_2 = \alpha \otimes \beta$. This means first of all that the simple vectors $\alpha \otimes \beta \in \mathfrak{S}$ coincide with the vectors of the form $w_1 \overline{\otimes} w_2$

† In the previous section we already introduced rules for the special cases; the rules stated here are simply their instances for the situation at hand. If we have restated them here, it is only in order to make the definition of $\overline{\otimes}$ more self-contained.

♭ One can also deduce the bilinearity of $\overline{\otimes}$ directly from the definition (16), if one proves first the following useful lemma: The set of first rows $(a, b)$ which make a square with a fixed second row $(c, d)$ is closed under addition and scalar multiplication; and similarly for columns instead of rows. Closure under scalar multiplication we already noticed, and closure under sum can be deduced from the general square-form (15). Taken together, the row and column assertions suffice to prove linearity of $\overline{\otimes}$ in both arguments.

for some $w_i \in W_i$ $(i = 1, 2)$. Consequently, we can build up a basis of $V$ by choosing a basis $\{e_j \mid j = 1 \cdots \dim W_1\}$ for $W_1$, and a similar basis $\{f_k \mid k = 1 \cdots \dim W_2\}$ for $W_2$, and then taking our basis-elements to be $e_{jk} = e_j \overline{\otimes} f_k$. That these $e_{jk}$ constitute a basis for $V$ follows from the fact that the $e_j$ (respectively the $f_k$) have the form $\alpha_j \otimes \beta_0$ (resp. $\alpha_0 \otimes \beta_k$), whereby the $\alpha_j$ (resp. $\beta_k$) constitute a basis of $V_1$ (resp. $V_2$) if and only if the $e_j$ (resp. $f_k$) constitute a basis of $W_1$ (resp. $W_2$), and furthermore $e_{jk} = e_j \overline{\otimes} f_k = \alpha_j \otimes \beta_k$.

The upshot is that a "special basis" for $V$ (i.e. a basis of vectors $\alpha_j \otimes \beta_k$) is the same thing as a pair of bases for $W_1$ and $W_2$, modulo the familiar $GL(1)$ ambiguity that one can rescale the $W_1$-basis by $\lambda$ if one simultaneously rescales the $W_2$-basis by $1/\lambda$. Recall now from Section 2 that our "second possible answer" to what constitutes a tensor-product structure for $V$ was "an equivalence-class $\mathfrak{T}$ of special bases for $V$". We have thus demonstrated that from $\mathfrak{S} \subseteq V$ one can derive uniquely a tensor-product structure in that sense. Conversely, given such a structure $\mathfrak{T}$, we immediately obtain $\mathfrak{S}$ from it as the union of all of the members of the special bases that comprise $\mathfrak{T}$. To the extent that $\mathfrak{S}$ is a simpler and more natural object than an equivalence class of special bases (and is also more intrinsic to $V$), we have reason to maintain that in $\mathfrak{S}$ we have an answer to the question, "What does it mean for $V$ to be a tensor product?".

### *Using "pointed vector spaces"*

The above construction began with an arbitrarily chosen "base-vector" $w_0 \in V$ such that $W_1 = \pi_1 w_0$ and $W_2 = \pi_2 w_0$. The ambiguity inherent in such a choice does not impugn our demonstration of the equivalence, $\mathfrak{T} \leftrightarrow \mathfrak{S}$, but it does mean that in the procession, $(V_1, V_2) \to V\text{-cum-}\mathfrak{S} \to (W_1, W_2)$, a different choice of $w_0$ would produce a different pair of spaces, $W_1$, $W_2$. If desired, one could arrange for $W_1$ and $W_2$ to be unique by working with "pointed vector spaces", i.e. by equipping $V_1$ and $V_2$ with distinguished "base-points", $\alpha_0 \in V_1$ and $\beta_0 \in V_2$, and then taking $v_0 = \alpha_0 \otimes \beta_0 \in V_1 \otimes V_2$ to be the base-point of $V$. Our construction above (with $w_0$ taken to be $v_0$) would then recover the pairs $(V_1, \alpha_0)$ and $(V_2, \beta_0)$ essentially uniquely from the triple $(V, \mathfrak{S}, v_0)$.

Remark Interestingly, the "histories Hilbert spaces" $\mathfrak{H}$ that play a role in Quantum Measure Theory [3] automatically come with distinguished vectors $|\Omega\rangle \in \mathfrak{H}$, where $\Omega$ represents the full history-space (the unit of the corresponding event-algebra). However it is generally false for coupled subsystems that $\mathfrak{H}$ for the composite system is the Hilbert-space tensor product of the $\mathfrak{H}$'s for the subsystems. (Even when the vectorspace dimensions match, the norms in general will not.)

## 6. Categorical matters (and a shortcoming of the functor concept)

From a given vector space $V$ one can form new spaces, like the dual-space $V^*$ or the double dual $V^{**}$. With two vector spaces, there are other possibilities, including their direct sum, their tensor product, and so forth. Although a vector space formed in one of these ways will be isomorphic to infinitely many other vector spaces, its "inner constitution" will in general be distinctive, with the result that it will to a certain extent "remember where it came from". One may say then that it *carries the structure of* a dual space, a direct sum, or a tensor product, as the case may be. In each instance one can try to identify concretely where this extra information resides, and for a vector-space $V$ that arose as a tensor product, our discussion has pointed to the set $\mathfrak{S}$ of simple vectors within $V$ as the pertinent structure. Adopting a notation that keeps track of $\mathfrak{S}$, we may say that from an ordered pair $(V_1, V_2)$ of vector spaces, there arises via tensor-product the ordered pair $(V, \mathfrak{S})$.

A question then is to what extent the transformation $(V_1, V_2) \to (V, \mathfrak{S})$ is reversible. How perfectly does $V$ remember where it came from, or to ask this another way, how well can we reconstruct $V_1$ and $V_2$, given $V$ and $\mathfrak{S}$? When we dealt with pointed spaces, we discovered that $(V_1, V_2)$ could "in essence" be recovered fully. But in the unpointed case, it appeared that although $(V, \mathfrak{S})$ is determined by $(V_1, V_2)$, the latter could be recovered from the former only up to some sort of $GL(1)$ ambiguity. This suggests that in the pointed case a vector space carrying the structure of a tensor product is in some sense equivalent to the factor spaces from which it arose, whereas in the un-pointed case there is only partial equivalence.

But what concept of equivalence is implicitly animating these expectations? Simple isomorphism will not do, being too narrow in one way (because by definition structures of different types like $(V_1, V_2)$ and $(V, \mathfrak{S})$ cannot be isomorphic) and too broad in another way (because, for example, any two vector spaces of equal dimension are isomorphic). Maybe one can put the underlying thought into words by saying that "$A$ and $B$ are equivalent if $B$ can be constructed from $A$ and vice versa." I am not sure that mathematics knows any framework which really does justice to this thought, but perhaps the category-theoretical concepts of functor and natural isomorphism come closest to providing one, and so it seems worth considering how they apply to the question at hand. We will take this up momentarily, but first let's see a very simple illustration of how $\mathfrak{S}$ is able to remember "where $V$ came from".

### *A small illustration: topology remembers spin*

As a small illustration of how the set $\mathfrak{S}$ of simple tensors encodes the structure of $V$ as a tensor product, let us return to the example of *spin*-3/2 $\otimes$ *spin*-1 versus *spin*-1/2 $\otimes$ *spin*-5/2. To distinguish these two possible provenances of $V$, one from the other, it is enough to pay attention to the topology of $\mathfrak{S}$, for example its dimensionality. Taking into account that an element of $\mathfrak{S}$ has by definition the form $\alpha \otimes \beta$, and that $\alpha$ and $\beta$ are unique modulo the obvious $GL(1)$ ambiguity, we can observe that the (complex) dimensionality of $\mathfrak{S}$ is one less than the sum of the dimensionalities of the factor spaces. In our examples this yields for $\dim(\mathfrak{S})$ the respective values, $4+3-1=6$ for $\mathbf{3/2} \otimes \mathbf{1}$ and $2+6-1=7$ for $\mathbf{1/2} \otimes \mathbf{5/2}$. In fact, it is easy to verify that this simple test works in general. If we know that $V$ arose from the combination of two spins, then the topological dimension of $\mathfrak{S}$ determines fully what those spins were. Of course (and as we have now seen in great detail) the same information can be deduced with a bit more work from the dimensionalities of the maximal linear subspaces of $\mathfrak{S}$, or from the dimensionalities of $\mathcal{M}_1$ and $\mathcal{M}_2$ as "foliations" of $\mathfrak{S}$.

### *A functorial gloss on our constructions*

Now back to categories, functors, and natural isomorphisms. A *category* is basically a collection of spaces of a given type (its "objects") and of structure-preserving mappings between these spaces (its "morphisms"). For our purposes it will be best to limit the latter to isomorphisms, i.e. to require them to be invertible. A *functor* between two categories, I and II, is a kind of black box that converts the objects and morphisms of category-I to objects and morphisms of category-II while preserving composition of morphisms. Conceptually, it is telling you that you can build spaces and mappings of type II from spaces and mappings of type I (but unfortunately it is not telling you *how* to do so.)

The two categories of interest to us here can be denoted as VEC $\times$ VEC and TVEC, where the former is the category of pairs $(V_1, V_2)$ and the latter $^\star$ is the category of pairs $(V, \mathfrak{S}_V)$. A morphism in VEC $\times$ VEC will thus be a pair of linear isomorphisms, while a morphism in TVEC will be a linear isomorphism between vector spaces that preserves their respective subsets $\mathfrak{S}$. When our spaces are pointed, all these isomorphisms will of course also need to preserve the respective base-points. Let us now describe some of our

$^\star$ The "T" in TVEC is meant to suggest the word "tensor" constructions in terms of functors between VEC $\times$ VEC and TVEC, concentrating for the time being exclusively on the pointed case.

The first functor of interest, which we will designate as $\otimes :$ VEC $\times$ VEC $\to$ TVEC, is that induced by the tensor product itself. It takes a pair of vector spaces $(A, B)$ to their tensor product, $V = A\otimes B$ equipped with its space $\mathfrak{S}_V$ of simple vectors $\alpha\otimes\beta$, and it takes a pair $(f, g)$ of (invertible) linear functions between vector spaces to their tensor product $f\otimes g$. Conversely, given an object $(V, \mathfrak{S}_V, v_0) \in$ TVEC (where I've now indicated the base-point $v_0$ explicitly), we saw how to locate within $V$ the subspaces $W_1 = \pi_1(v_0)$ and $W_2 = \pi_2(v_0)$, which were certain maximal linear subsets of $\mathfrak{S}_V$. Thereby, we in effect defined a second functor, $D :$ TVEC $\to$ VEC $\times$ VEC, that goes in the direction opposite to $\otimes$, and for which $D(V, \mathfrak{S}_V, v_0) = ((W_1, v_0), (W_2, v_0))$. Of course one has not defined a functor fully until one tells how it acts on morphisms, but that is self-evident for $D$. A morphism in TVEC from $(V, \mathfrak{S}_V, v_0)$ to $(V', \mathfrak{S}'_V, v'_0)$ is nothing but an invertible linear function, $f : V \to V'$, such that $f[\mathfrak{S}_V] = \mathfrak{S}'_V$ and $f(v_0) = v'_0$. Such an $f$ induces immediately a pair of (basepoint preserving) functions $f_1 : W_1 \to W'_1$ and $f_2 : W_2 \to W'_2$, and so $D(f) = (f_1, f_2)$.

Now what of the expectation that $\otimes$ and $D$ are in essence each other's inverses? Were that literally true, we would be able to express it by writing $\otimes \circ D = 1$ and $D \circ \otimes = 1$, but unfortunately both equations are, strictly speaking, false. Consider first the composed functor, $D \circ \otimes$. What happens when we apply it to the pair of pointed vector spaces $((A, \alpha_0), (B, \beta_0))$? Tracing through the definitions, we find $\otimes((A, \alpha_0), (B, \beta_0)) = (A\otimes B, \mathfrak{S}_{A\otimes B}, \alpha_0\otimes\beta_0)$, and then $D(A\otimes B, \mathfrak{S}_{A\otimes B}, \alpha_0\otimes\beta_0) = ((W_1, w_1), (W_2, w_2))$, where $W_1 = \pi_1(\alpha_0\otimes\beta_0) = A\otimes\beta_0$, $W_2 = \pi_2(\alpha_0\otimes\beta_0) = \alpha_0\otimes B$, and $w_1 = w_2 = \alpha_0\otimes\beta_0$. In other words,

$$(D \circ \otimes)((A, \alpha_0), (B, \beta_0)) = ((A\otimes\beta_0, \alpha_0\otimes\beta_0), (\alpha_0\otimes B, \alpha_0\otimes\beta_0)) \tag{17}$$

While $(D\circ\otimes)((A, \alpha_0), (B, \beta_0))$ is thus not exactly identical with $((A, \alpha_0), (B, \beta_0))$, there is between them an obvious correspondence, $((A, \alpha_0), (B, \beta_0)) \longleftrightarrow (D \circ \otimes)((A, \alpha_0), (B, \beta_0))$, given by the linear isomorphisms,

$$\alpha \longleftrightarrow \alpha \otimes \beta_0 \qquad \text{and} \qquad \beta \longleftrightarrow \alpha_0 \otimes \beta \tag{18}$$

The bijection (18) is an instance of what is called a *natural isomorphism* between functors, and so category theory gives us a precise way to express that $D$ is effectively a right-inverse of $\otimes$ by saying that $\otimes \circ D$ is "naturally isomorphic" to the identity-functor, a relationship which we will write as

$$\otimes \circ D \cong 1 \ . \tag{19}$$

It is evident from its definition that the correspondence (18) establishes an isomorphism between two *objects* in VEC $\times$ VEC. If read from left to right, it is a mapping $\Psi : ((A, \alpha_0), (B, \beta_0)) \to (D \circ \otimes)((A, \alpha_0), (B, \beta_0))$, but what is it that earns $\Psi$ the title, "natural", thereby authorizing the use of the symbol $\cong$ in (19)? It is that $\Psi$ also induces the correct correspondence between *morphisms* by converting $(f_1, f_2)$ into $(D \circ \otimes)(f_1, f_2)$. This is self-evident when one unpacks the definitions (cf. (20) below), but even without unpacking the definitions, we could have been assured that $\Psi$ was natural, if we had reflected that it was defined *intrinsically*, utilizing nothing more than the structures displayed in (17). Indeed, I think it would be fair to say that this possibility of being constructed from intrinsic information without the intervention of any arbitrary choices is what best expresses the intuitive meaning of "naturality".

The distinction between plain isomorphism $\simeq$ and natural isomorphism $\cong$ is perhaps most familiar in the example of dual vector-spaces, where both $V^{**}$ and $V^*$ are isomorphic to $V$, but only the isomorphism between $V^{**}$ and $V$ is natural. Given an element $v \in V$ one can define $v^{**} \in V^{**}$ by the equation, $v^{**}(f) = f(v)$, where $f \in V^*$. On the other hand, there is no way to pass deterministically from $v$ to an element $f \in V^*$ without the aid of a basis for $V$, or a metric, or some such auxiliary structure.

Remark A natural isomorphism sets up an equivalence between functors in much the same way as a similarity transformation sets up an equivalence between group representations. If $R_1$ and $R_2$ are representations of the group $G$ related by the similarity transformation $S$, then $SR_1(g)S^{-1} = R_2(g)$, or equivalently $SR_1(g) = R_2(g)S$. Now if we replace $R_1$ and $R_2$ by functors $F_1$ and $F_2$, and the arbitrary group-element $g$ by an arbitrary morphism $f$, we obtain the condition for a family of invertible morphisms $S$ to define a natural isomorphism between $F_1$ and $F_2$, namely $SF_1(f) = F_2(f)S$. Often this last equation is represented by drawing the commutative diagram,

$$\begin{array}{ccc} F_1 X & \stackrel{F_1 f}{\to} & F_1 Y \\ S_X \downarrow & & \downarrow S_Y \\ F_2 X & \underset{F_2 f}{\to} & F_2 Y \end{array}$$

where $X$ and $Y$ are any objects in the category and $f : X \to Y$ is any morphism between them. When, as in our case, $F_1$ is the identity functor, the diagram for $S : 1 \to F$ simplifies to

$$\begin{array}{ccc} X & \overset{f}{\to} & Y \\ S\downarrow & & \downarrow S \\ FX & \underset{Ff}{\to} & FY \end{array} \tag{20}$$

One sees in this case that the functor $F$ must be a bijection between the morphisms $f$ and the morphisms $Ff$; and conversely, the fact that $F$ is such a bijection captures to a large extent everything that the equation $F \cong 1$ means.

Having established that $D \circ \otimes \cong 1$, let us now try to demonstrate the complementary equivalence, $\otimes \circ D \cong 1$. Following the same steps as before, let us apply the functor $\otimes \circ D$ to the object $(V, \mathfrak{S}_V, v_0) \in \text{TVEC}$, obtaining first $D(V, \mathfrak{S}_V, v_0) = ((W_1, v_0), (W_2, v_0))$ and then $\otimes((W_1, v_0), (W_2, v_0)) = (W_1 \otimes W_2, \mathfrak{S}_{W_1 \otimes W_2}, v_0 \otimes v_0)$, which taken together tell us that

$$(\otimes \circ D)(V, \mathfrak{S}_V, v_0) = (W_1 \otimes W_2, \mathfrak{S}_{W_1 \otimes W_2}, v_0 \otimes v_0) \tag{21}$$

Can we exhibit a natural isomorphism equating $(W_1 \otimes W_2, \mathfrak{S}_{W_1 \otimes W_2}, v_0 \otimes v_0)$ to $(V, \mathfrak{S}_V, v_0)$, and therefore $\otimes \circ D$ to the identity functor? To this question we already have the answer in the form of the isomorphism, $\Phi : W_1 \otimes W_2 \to V$, which we constructed earlier with the aid of the intrinsically defined product $\overline{\otimes}$, and for which $\Phi(w_1 \otimes w_2) = (w_1 \overline{\otimes} w_2)$. As with $\Psi$ before, it is straightforward to verify that $\Phi$ is natural, as indeed it had to be, given its intrinsic nature. Therefore $\otimes \circ D \cong 1$.

We have now proven that the composition of $\otimes$ with $D$ in either order is naturally isomorphic to the identity. Thus category theory, by introducing the concept of natural isomorphism $\cong$ as a replacement for strict equality, has given us a way to make precise (and then to verify) the informal claims that, in the pointed case, the functor $\otimes$ is invertible and that $D$ is its inverse.

Turn now to the unpointed case and to our expectation that it will not be possible to recover the pair $(A, B)$ from $(A \otimes B, \mathfrak{S})$ when the spaces involved are not equipped with base-points. Can we also corroborate this expectation within the categorical framework? Stated formally, the question is whether there exists a functor $D : \text{TVEC} \to \text{VEC} \times \text{VEC}$ which is a "left inverse" to $\otimes$ in the sense that $D \circ \otimes \cong 1$. In fact, it's easy to see that no such functor can exist. Were $D$ such a functor then, as we noticed in connection with (20), the mapping, $f \mapsto (D \circ \otimes) f$, would have to be invertible for morphisms, $f : (A, B) \to (A', B')$, of the category $\text{VEC} \times \text{VEC}$, where $f$ is by definition a pair $(g, h)$ of individual morphisms in VEC. This, however, is clearly impossible because the functor $\otimes$ (and therefore its composition with $D$ if the latter existed) fails to be injective, since it maps both $f = (g, h)$ and $\tilde{f} = (\lambda g, h/\lambda)$ to the single morphism $g \otimes h = (\lambda g) \otimes (h/\lambda)$. In other words, $\otimes$ acting on morphisms is not injective but many-to-one, the "many" being parametrized by a non-zero scalar $\lambda$ which embodies the same $GL(1)$ ambiguity we met with earlier. This confirms that the equation $D \circ \otimes \cong 1$ can have no solution, and a fortiori that the functor $\otimes$ is not invertible.

A somewhat simpler example of the same nature occurs in connection with the attempt to represent a spinor geometrically. Starting from a 2-component Weyl spinor $\zeta$, for example, one can derive algebraically a so-called null flag $F$, which consists of a light-like vector together with a half-plane matched to the vector. [4] But because vectors are quadratically related to spinors, both $\zeta$ and $-\zeta$ give rise to the same flag $F$, whence one can recover the spinor from the flag only up to an unknown sign. (This loss of information was inevitable, because spinors change sign after rotating through $2\pi$, whereas vectors do not.) To couch these relationships in categorical language, one could introduce a category of spinor-spaces and a category of spaces of null-flags and a functor $\phi$ from the former to the latter. Like $\otimes$ above, $\phi$ would not be invertible, because it would be $2 \to 1$ on morphisms. At best, one might be able to devise, as a kind of *right* inverse to $\phi$, a "functor manqué" or "functor up to sign" going from flag-spaces to spinor spaces. Its existence would proclaim that, although not fully a geometrical object, a spinor is nevertheless "geometrical up to sign".

REMARK Despite its utility, the concept of functor does not necessarily illuminate the connection between its inputs and its outputs as fully as one might have expected it to do, because unlike a morphism, it is blind to the individual elements of the spaces on which it acts; by definition it does not "look inside". Thus if $\phi$ is a functor and $X$ a space (or a mapping), and if $Y$ is the space (or mapping) that results when $\phi$ acts on $X$, then the equation $Y = \phi(X)$ tells us *that* $Y$ is in some sense built from $X$, but it tells us nothing concretely about *how* $Y$ is built from $X$.† For example $X$ could be a spinor-space, and $\phi$ the above functor. Then $Y = \phi(X)$ would be the space comprised of all the null flags derived from the spinors comprising $X$. But if $\zeta \in X$ were an *individual* spinor in $X$, and if $F$ were the individual flag derived from $\zeta$, the rules governing functors would not allow

† Could Bourbaki's concept [5] of "deduction of structures" come any closer to doing this?

us to write “$F = \phi(\zeta)$”, even though it might seem natural to do so, and even though we know perfectly well what we would mean by it!

## 7. Questions; further developments; connection to quantum field theory

In conclusion, let me mention a few questions and possible further developments suggested by the above considerations.

The most important of the constructions introduced in Sections 4 and 5 revolve around the “foliations” $\mathcal{M}_1$ and $\mathcal{M}_2$, the corresponding mappings $\pi_1$ and $\pi_2$, the *square* concept, and the product $\overline{\otimes}$ which results from these via the definition (16).

An obvious question that one might ask is how these distinctive ingredients generalize to the tensor product of three or more vector spaces. One could of course just treat a threefold product like $A \otimes B \otimes C$ as an iterated pairwise product like $(A \otimes B) \otimes C$, but a more symmetric construction ought to be possible, and one might expect it to uncover some new structures that are not visible in connection with simple pairwise products like $A \otimes B$.

One might also wonder whether there was anything of interest to be learned from the study of the various symmetry types that become possible when two or more of the factor-spaces are equal to each other. For example when $V \subseteq A \otimes A$ is the subspace of symmetric tensors, two natural analogs of $\mathfrak{S}$ as used above would be the set of tensors of the form, $\alpha \otimes \beta + \beta \otimes \alpha$, or even more simply, of the form $\alpha \otimes \alpha$. Or for the anti-symmetric tensor product of $A$ with itself, the set of tensors of the form, $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$, would be a natural analog of $\mathfrak{S}$. To what extent, and in what form, could one repeat the above discussion with one of these subsets replacing $\mathfrak{S}$? And still more generally, what might be an analog of $\mathfrak{S}$ belonging to the non-abelian symmetry-types (those corresponding to more general Young tableaux) that arise as subspaces of higher products, $A \otimes A \cdots \otimes A$, and which are neither “bosonic” nor “fermionic”?

Another sub-case of obvious interest is that where the vector spaces are equipped with metrics, in particular where they are Hilbert spaces. One might then expect orthonormality to play a role, but would any additional, unexpected features of interest show up?

Our discussion so far has proceeded in finite dimensions. If we want to generalize it to infinite-dimensional vector spaces, a whole raft of further questions will appear, some of which concern the definition of tensor product itself. Clearly, the subset $\mathfrak{S}$ of simple tensors $\alpha \otimes \beta$ within $V = A \otimes B$ can be defined without difficulty, but will our constructions based on it also go through as before? Will they still let us recover $A$ and $B$, and will they still lead us as in Sections 4 and 6 (say in the pointed case) to a functor $D$ inverse to $\otimes$? In all of this, what consequences might flow from ambiguities in the definition of $\otimes$? When $A$ and $B$ are Hilbert spaces, $A \otimes B$ qua Hilbert space is unambiguous, but when they are only Banach spaces (normed vector spaces), many different spaces $A \otimes B$ have been defined [2]. One may wonder then whether $\mathfrak{S}$ will still be able to "remember" which specific choice of $\otimes$ went into the creation of $V$.

Among infinite-dimensional vector spaces, the Hilbert spaces have a special significance for quantum theories. Although the ambiguity in defining $A \otimes B$ is not an issue when $A$ and $B$ are Hilbert spaces, it can happen in connection with quantum field theory that the notion of tensor product itself seems to be transcended. If one divides a Cauchy surface into two complementary regions, then naively one would expect the overall Hilbert space $\mathfrak{H}$ of the field-theory to be the tensor product of Hilbert spaces associated with the two regions, just as happens with composite systems in ordinary quantum mechanics. Unfortunately, this would conflict with the fact that the operator algebras associated with the two regions (technically with their domains of dependence in spacetime) are known (for free fields) to be of "type III", this being intimately linked to the infinite entanglement-entropy between the two regions. One still has operator subalgebras for the regions (so-called coupled factors), but these subalgebras cannot be interpreted as acting on the separate factors of a tensor-product. One thus confronts something like a tensor product of operator-algebras that does not derive from a tensor-product decomposition of the underlying Hilbert space $\mathfrak{H}$. (Adopting the language of "quantum systems", one might say that one is dealing with "subsystems which possess observables but lack state-vectors".) $^\flat$

In the absence of a tensor-product structure for $\mathfrak{H}$, the notion of simple-vector is not defined, and therefore neither is our subset $\mathfrak{S} \subseteq \mathfrak{H}$. Nevertheless, one might hope that

$^\flat$ This is not quite the same as saying that a type-III factor lacks pure states. As most commonly defined in the theory of operator-algebras, a pure state on $\mathfrak{A}$ is an extreme point in the convex set of normalized positive linear functionals on $\mathfrak{A}$. It is known that such pure states exist copiously, and one could thus entertain them as generalized state-vectors, since in finite dimensions, state-vector = pure state. However when $\mathfrak{A}$ is a type-III factor, its pure states seem to be mathematically pathological (perhaps even "ineffable"), and one could plausibly regard them as unphysical. See [6].

some generalization of simple vector, and some corresponding subset of $\mathfrak{H}$, could serve a similar function. Simple vectors are tensors of rank 1, but the tensors of ranks, 2, 3, 4, etc. also respond to the tensor-product structure of $V$. Could it be that suitable analogs of the spaces of such tensors (or better, of the tensors of finite "co-rank" in some suitable sense) are able to capture the structure of coupled factors of types-III or II, and in so doing shed light on features like the area-law for entanglement-entropy? Especially salient in this connection is the "spatiotemporal cutoff" needed to render the entropy finite [7]. Physically, such a cutoff needs to be frame-independent (locally Lorentz invariant), and it seems suggestive that an analog of $\mathfrak{S}$, if it could be defined, would not obviously need to refer to any arbitrarily chosen reference-frame.

As a first approach to some of these questions, one could ask in finite dimensions how to relate the operator-algebra framework to that of the present paper. Indeed, one might have thought to identify a tensor-product-structure for $V$, not with the family $\mathfrak{S}$ of simple vectors in $V$, but with a pair of commuting operator-subalgebras which generate the algebra $L(V)$ of all linear operators on $V$ and which have in common only the multiples of the identity-operator (like Murray-von Neumann coupled factors but without any specialization to complex numbers or self-adjointness). The advantage of such an alternative approach would be that the algebras $L(A)$ and $L(B)$ reappear bodily in $L(A \otimes B)$, whereas the spaces $A$ and $B$ themselves need to be excavated from $A \otimes B$ more painfully, as we have seen in great detail above. (Quantally speaking, the "observables" of a subsystem carry over to the composite system, whereas the "states" do not. But see the remark below.) Its disadvantage would be that an algebra of operators in $V$ is a considerably more complicated beast than the simple subset $\mathfrak{S} \subseteq V$. Be that as it may, it's clear that the two viewpoints are related. For example, an operator acting only on $A$ (an operator in $L(A) \otimes \mathbf{1}$), or only on $B$, will automatically be an operator that preserves $\mathfrak{S}$, suggesting how one might derive $L(A)$ and $L(B)$ from $\mathfrak{S}$.

REMARK In the context of Quantum Measure Theory [3], the histories-hilbert-space associated to a subsystem actually does reappear as a true subspace of the histories-hilbert-space of the full system, the reason being that an *event* in a subsystem is *ipso facto* an event in the full system. Moreover this subspace carries a distinguished "base point", as remarked in Section 5. When, in addition, the overall quantum-measure is the product-measure (as for "non-interacting subsystems in a product-state"), the full histories-hilbert-space is the tensor product of the subspaces, and the aforementioned advantage of an approach via operator algebras disappears.

Let us return, finally, to finite dimensions and to the cone $\mathfrak{S}$ of simple vectors within $V$, on which most of our work has been based. We have seen how $\mathfrak{S}$ endows $V$ with the structure of a product space, but we did not provide (or even ask for) a simple criterion that would let us recognize whether a given subset $\mathfrak{S}$ could actually play the role assigned to it. That is, we did not provide necessary and sufficient conditions for there to exist an isomorphism mapping $V$ to a space $A \otimes B$ that would map $\mathfrak{S}$ to the set of tensors of the form $\alpha \otimes \beta$. One trivially adequate criterion is that the re-constructions undertaken in Section 5 should succeed, and in particular that the building up of the *squares* should never encounter an obstacle. But one might wish for criteria that were more self-contained and more simply stated. Given that the rays in $\mathfrak{S}$ constitute a "Segre variety", one might hope that the Algebraic-Geometry literature would contain something of this sort.

Alternatively, rather than seeking axioms for $\mathfrak{S}$, one might instead seek axioms for the squares themselves, i.e. axioms for quadruples of vectors in $V$. A tensor-product structure for $V$ would then be a set of quadruples satisfying these axioms.

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Chapter 1 of Sternberg contains examples of most of the definitions, including implicitly the definition of $V \otimes W$ as the space of bilinear mappings of $V^* \times W^*$ into $\mathbb{R}$. The time-honoured definition of a tensor in terms of transformation laws for its components is presented in Thomas.

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