On the Entropy of the Vacuum outside a Horizon[⋆]

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The evidence is very strong that a black hole presents itself to the outside world as a thermodynamic system with entropy proportional to its surface (horizon) area. Yet the physical origin of this entropy is far from clear. In fact the formula $S = k \lg N$, on which our general understanding of the Second Law is based, entails the absurdity $S = \infty$; for—unlike in flat space—a bound on the total energy does not suffice to bound the number of possible internal states. In particular the Oppenheimer-Snyder solutions [1] already provide an infinite number of possible internal configurations for a Schwarzschild exterior of fixed mass.

A related observation is that the internal dynamics of a black hole ought to be irrelevant to its exhibited entropy because—almost by definition—the exterior is an *autonomous system* for whose behavior one should be able to account without ever referring to internal black hole degrees of freedom. In particular one should be able to explain why it happens that a sum of two terms, one referring to exterior matter and the other only to the black hole geometry, tends always to increase.

Based on the conception of the exterior region as an autonomous quantum system with state given by the density matrix, $\rho = \rho^{ext}$, one can automatically define an "exterior

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entropy" $S^{ext} = -\operatorname{tr} \rho \lg \rho$. This paper will estimate a particular contribution to S^{ext} and show that it does in fact produce a term proportional to the horizon area. The other half of the problem—showing that S^{ext} increases—will not be addressed (except to point out herewith that on general grounds it suffices to show that the "totally random" state $\rho = 1$ evolves into itself. [2])

The contribution we will consider pertains to the quantum fields assumed to exist in spacetime (including gravitons of course) but will be estimated only for a non-interacting scalar field. Specifically consider a Klein-Gordon field ϕ and a Cauchy hypersurface H^{tot} divided into two regions H^{int} and H^{ext}, and let the quantum (mixed) state of ϕ with respect to H^{tot} be ρ^{tot}. Tracing out the variables referring to H^{int} produces a reduced operator $\rho = \rho^{ext}$ which is the effective state for observers confined to H^{ext}. The entropy $S = S^{ext} = -\operatorname{tr} \rho \lg \rho$ is then defined and is in general non-zero even when ρ^{tot} itself is a pure state (in which case S is an intrinsically quantum entropy.) Moreover S depends not at all on the analogously reduced operator ρ^{int} describing the state of ϕ relative to the *interior* region H^{int}. Let us estimate S in the situation that H^{tot} is a t =constant hypersurface in Minkowski space and $\rho^{tot} = |0\rangle\langle 0|$ is the (Minkowski) ϕ-vacuum. (This sounds very unlike the situation of physical interest, but in fact turns out to yield a decent approximation to the latter.)

To begin with we can replace ϕ by a lattice of harmonic oscillators distributed in H^{tot} with density ℓ^{-3} and whose hamiltonian is a discretized version of $\int \frac{1}{2}[(\partial\phi)^2 + m^2\phi^2]d^3x$. With the lattice points labelled by an index A and ϕ^A the value of ϕ at the A^{th} such point, the Hamiltonian takes the form

$$\frac{1}{2}G^{AB}p_A p_B + \frac{1}{2}V_{AB}\phi^A\phi^B$$

where G and V are both positive definite matrices depending on the choice of lattice, and in the Schrödinger representation $p_A = -i\partial/\partial\phi^A$. Then with the vacuum $|0\rangle$ defined as the minimum energy state, a calculation whose details will appear elsewhere expresses $S = S^{ext}$ as a sum over the eigenvalues of a certain operator Λ: $S = \sum_\lambda S(\lambda)$, where, with μ abbreviating $1 + 2\lambda^{-1} - 2[\lambda^{-1}(1-\lambda^{-1})]^{1/2}$, $S(\lambda) := -\lg(1-\mu) - [\mu/(1-\mu)]\lg\mu$.

Λ itself depends on the division of the lattice points into interior (labelled by α, β) and exterior (labelled by a, b) and can be expressed as $\Lambda^a{}_b = -W^{a\alpha} W_{\alpha b}$ where W^{AB} is the inverse matrix of W_{AB}, which in turn is the positive square root of V_{AB} with respect to the scalar product G^{AB}: $W_{AB} G^{BC} W_{CD} = V_{AD}$. (In the continuum limit W_{AB} is the integral operator whose kernel is the so-called "finite part" of $-\pi^{-2} |x-y|^{-4}$.)

On dimensional grounds it is easy to see that S will be ultra-violet infinite in the continuum limit $\ell \to 0$. For finite ℓ the fact that S is a *sum* over the eigenvalues of Λ, coupled with the fact that the singularity in $\Lambda = \Lambda(x,y)$ occurs at $x = y$, means that the leading term in S will be proportional to A/ℓ^2 where A is the area of the surface ("horizon") separating H^{int} from H^{ext}. The proportionality constant can be estimated as $3 \int_0^1 y\, dy\, \sigma(y^2 + m^2 \ell^2)$ where $\sigma(y^2)$ is the value of S^{ext} computed in $1+1-$dimensions with the mass equal to y in units of the inverse lattice spacing and with H^{ext} taken to be a half-line. (Since for $y \to 0$ we have $-y\sigma(y^2) \sim y \lg y \to 0$, S is to leading order independent of m, as one would expect.)

To obtain an entropy of the correct order of magnitude for a black hole, the cutoff ℓ must be chosen approximately equal to the Planck length. Conversely, if S^{ext} really can be identified as the black hole entropy we obtain evidence of the physical necessity for such a cutoff to exist.

However caution is indicated by several circumstances in addition to the obvious one that it remains to be explained why S^{ext} necessarily increases with time. Aside from this, the most serious problem seems to be that for free fields S^{ext} would be proportional to the number of massless fields, ℓ being fixed. If this dependence is not cured by the presence of large couplings at high energies (and it may well not be in asymptotically free theories) or by a (supersymmetric?) conspiracy relating ℓ to the number and type of fundamental fields, then it must be cured by taking into account the coupling between the fields and the horizon shape itself ("back reaction"). This in any case must introduce an ℓ limiting the validity of the above semi-classical computation, and moreover limiting it in such a way as to tend to cancel the unwanted dependence on the number of fields. Beyond evaluating this cutoff (which should be done) one could try to take the next logical step of evaluating the degrees of freedom of the horizon itself, which one might argue also are about $\exp(A/\ell_{Planck}^2)$ in number. For if the entropy can't be inside the black hole and proves not to be outside it either, then where else can it possibly be but the horizon?

References

[1] Misner, C.W., K.S. Thorne and J.A. Wheeler (1970) *Gravitation*, San Francisco: W.H. Freeman

[2] Schnakenberg, J. (1976) Rev. Mod. Phys. **48** 571