October 25, 2018

XY Model

$$Z = \int \prod d\theta_{\vec{n}} \; \exp\left(\beta J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)\right)$$

low T

$$\simeq \int \prod d\theta_{\vec{n}} \; \exp\left(-\frac{\beta J}{2} \sum_{\langle ij \rangle} (\theta_i - \theta_j)^2\right)$$

large dist.

$$\simeq \int \mathcal{D}\vec{\theta}(\vec{x}) \; \exp\left(-\beta J \int (\vec{\nabla}\theta)^2 \, d^2\vec{x}\right)$$

$$\langle S_{\vec{y}} \cdot S_{\vec{y}+\vec{x}} \rangle = \langle e^{i\theta_y} e^{-i\theta_{x+y}} \rangle = e^{\cancel{G(\vec{x}) - G(0)}}$$

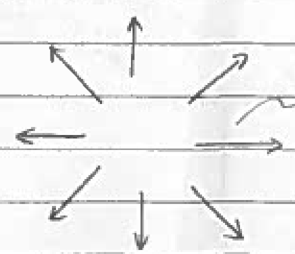
$$= e^{\frac{-1}{4\pi\beta J}\Gamma(\vec{x})}$$

$$\simeq \frac{1}{|\vec{x}|^{1/4\pi\beta J}} = \exp\left[\frac{-1}{4\pi\beta J} \log|\vec{x}|\right]$$

where $\Gamma(\vec{x}) = \displaystyle\int \frac{d^2k}{2\pi} \frac{e^{i\vec{k}\cdot\vec{x}}}{4\sin^2\left(\frac{k_x}{2}\right) + 4\sin^2\left(\frac{k_y}{2}\right)} \simeq \log\frac{r}{r_0}$

$$r_0 = \frac{1}{2\sqrt{2}\, e^{\gamma_E}}$$

Vortices:



$\theta \simeq n\phi$
$\uparrow$
charge

$$\left.\begin{aligned} -\beta E &= -\beta J \, 2\pi \int_{r_0}^{L} r\,dr \, \frac{1}{r^2} \simeq -2\pi\beta J \log L \\ S &\sim \log L^2 \quad (L^2 \text{ places to put the vortex}) \end{aligned}\right\} J\beta_c = \frac{1}{\pi}$$

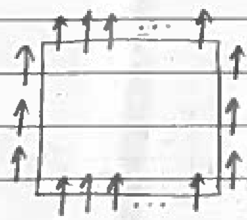
$$\left\{\begin{array}{l}\text{Charge } n = \pm 1: \\ \vec{\nabla}\theta = \left(\frac{y}{r^2}, \frac{-x}{r^2}\right) \\ (\vec{\nabla}\theta)^2 = \frac{x^2+y^2}{r^4} = \frac{1}{r^2}\end{array}\right\}$$

At $T = T_c$:

$$\langle SS \rangle \sim \frac{1}{|x|^{1/4}} \sim \exp\left(\frac{-1}{4} \log|x|\right)$$

Consider the boundary conditions:

If



Then inside we must have total zero charge.

Gas of vortices with $\sum_j n_j = 0$

Let us now consider the expansion:

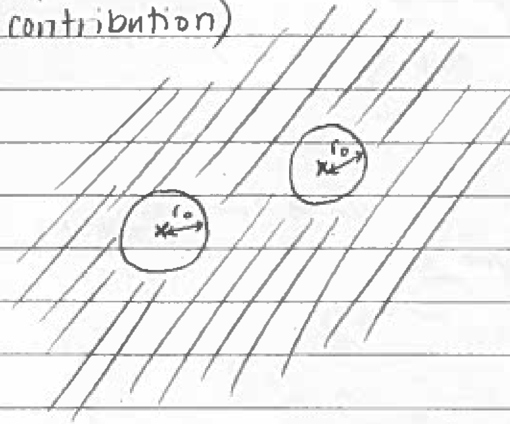
$$\theta(\vec{x}) = \underbrace{\theta_{\text{vortices at positions } \vec{r}_j}}_{\text{min. action, } \vec{\nabla}^2\theta = 0} + \theta_{\text{spin waves}}$$

fluctuations that don't change the vorticities

$$\theta = \sum_j n_j \arctg \frac{x - x_i}{y - y_i}$$

$$Z = Z_{\text{vortices}} \times \underbrace{Z_{\text{Gaussian}}}_{\text{spin waves}}$$

Introduce some vortex radius r_0 to regulate (positions within this radius give a finite contribution)



$$-\beta J \int d^2\vec{x}\,(\vec{\nabla}\theta_{\text{vortex}})^2 \simeq -S_{\text{vortex}}(n_j, \vec{r}_j)$$

$$-S_{\text{vortex}} = 2\pi\beta J \sum_{i\neq j}{}' n_i n_j \log\frac{|\vec{r}_i - \vec{r}_j|}{a}$$

lattice spacing $a=1$

$$+ 2\pi\beta J \sum_i{}' n_i^2 \log\frac{r_0}{a}$$

actually a-independent

$$\sum_i{}' n_i = 0$$

$$\sum_i{}' n_i \underbrace{\sum_{j\neq i}{}' n_j}_{=-n_i} = -\sum_i{}' n_i^2$$

n +1 vortices, n −1 vortices (leading contrib.) $\Rightarrow n_i^2 = 1$

$$\boxed{Z_{\text{vortices}} = \sum_{n=0}^{\infty}{}' \frac{K^n}{(n!)^2} \int_{|\vec{r}_i - \vec{r}_j|>a} d\vec{r}_1 \cdots d\vec{r}_{2n} \exp\left[2\pi\beta J \sum_{i\neq j}{}' n_i n_j \log\frac{|\vec{r}_i - \vec{r}_j|}{a}\right]}$$

where $K = \exp\left(2\pi\beta J \log\frac{r_0}{a}\right)$

Grand canonical Z for a gas of Coulomb charged particles in 2D

XY model:

no vortices — T_c — spin vortex config.

exp. decay

Coulomb:

dielectric material — T_c — plasma

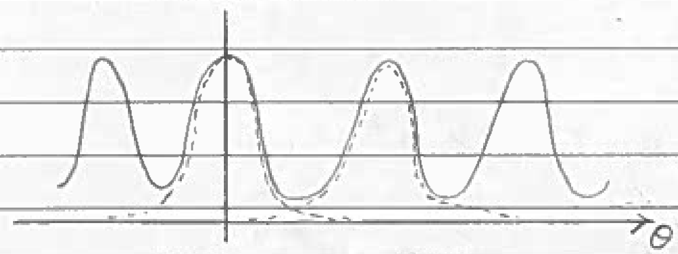
bunch of freely moving particles

screening

Now we will find yet another model in the same universality class:

— T_c — } vortices come out without thinking

$$Z = \int \prod d\theta_{\vec{n}} \prod_{\langle ij\rangle} \underbrace{e^{+\beta J \cos(\theta_{ij})}}_{z(\theta)}$$



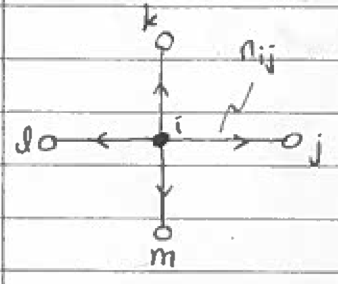
$\simeq \sum$ gaussians (similar figure & same $\theta \to 0$ behaviour)

Then using the Poisson summation formula (see Tutorial 6):

$$z(\theta) \simeq \sum_{n=-\infty}^{\infty} e^{\frac{-n^2}{2\beta J} + in\theta}$$

Bad: new d.o.f. n_{ij} on each link

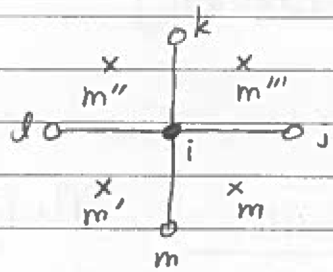
Good: can integrate over θ!



$$e^{i\theta_i(n_{ij}+n_{ik}+n_{il}+n_{im})} \xrightarrow[\int d\theta_i]{} 0$$

Now the partition function is just $\sum'_{n_{ij}}$ with $\sum'_{j} n_{ij} = 0$ (j n.n. of i)

Then we can define new d.o.f. at each face:



$n_{ij} = m - m'''$

etc.

$\vdots$

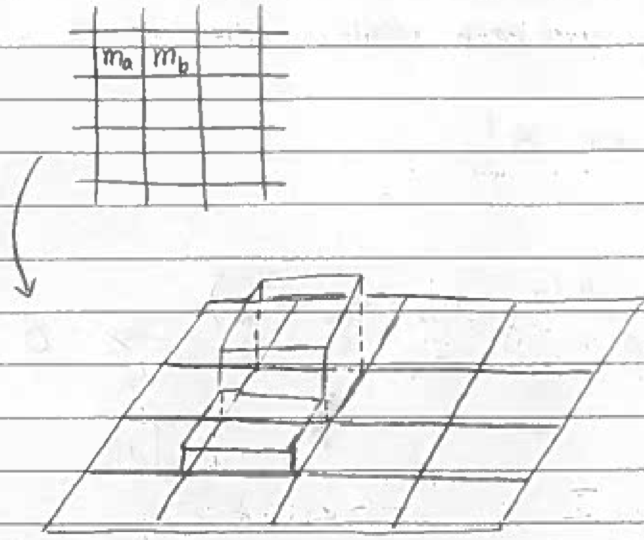
We find that:

$$Z_{XY} \underbrace{\simeq}_{\text{same universality class}} Z_{\mathfrak{z}(\theta) = \sum \text{Gaussians}}$$

$$= Z_{\text{SOS or fluctuating surface model}}$$

$$= \sum'_{\substack{m \in \mathbb{Z} \\ \text{per face}}} \exp\left[-\frac{1}{2\beta J} \sum'_{\langle ab \rangle} (m_a - m_b)^2\right]$$

Like the Gaussian limit of the XY model but $\beta \to \frac{1}{\beta}$



"Solid-on-solid" model (cubes on top of a surface)

- β small (T large): smooth (non-equal m's suppressed)

- β large (T small): rough (large fluctuations)

We have:

rough —— $\bullet$ —— smooth

T_c

fluctuating surface model

fluctuating surfaces: when Δm are large, it is not important that the m are integers
$\rightsquigarrow$ continuum limit

XY	SOS
• $O(2)$ symmetry	• Additive group of integers (can shift all m)
• lattice (cusps)	• dual lattice (faces)
• βJ	• $1/\beta J$

high-low temperature duality

Now, let's consider vertices in the SOS model

$$\sum_{m\in\mathbb{Z}} f(m) = \int d\varphi\, f(\varphi)\, \underbrace{\sum_m \delta(\varphi - m)}_{=\sum_{q\in\mathbb{Z}} e^{2\pi i q\varphi}}$$

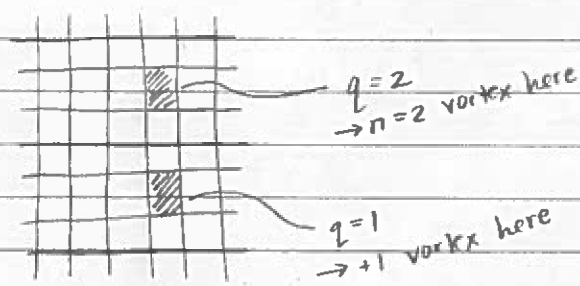
with $\sum q_{\vec{a}} = 0$ b/c of zero mode for $m \to m+$ integer

Gaussian!

$$\rightsquigarrow Z_{SOS} = \int \prod_{\text{face}} d\varphi_{\vec{a}} \sum_{q_{\vec{a}}}{}' \exp\left[\frac{-1}{2\beta J}\sum (\varphi_{\vec{a}} - \varphi_{\vec{b}})^2 + 2\pi i \sum q_{\vec{a}}\varphi_{\vec{a}}\right]$$

$$= Z_{SW} \sum_{q_{\vec{a}}} \exp\left[2\pi\beta J \sum_{\vec{a},\vec{b}} \vec{q}_{\vec{a}} \cdot \underbrace{\Gamma(\vec{r}_a - \vec{r}_b)}_{\log\frac{|\vec{r}_a - \vec{r}_b|}{r_0}} \cdot \vec{q}_b\right]$$

$$\boxed{Z_{SOS} = Z_{SW} \sum_{\substack{q_{\vec{a}} \text{ per face} \\ w\ \sum q_a = 0}} \exp\left[2\pi\beta J \sum_{\vec{a},\vec{b}} q_{\vec{a}} \log\frac{|\vec{x}_a - \vec{x}_b|}{a} q_{\vec{b}} + \pi\beta J \sum q_{\vec{a}}^2 \log\frac{r_0}{a}\right]}$$



Another model in the same universality class:

$$S = \int (\partial\varphi)^2 - 2\lambda\cos(2\pi\varphi) \qquad \text{Sine-Gordon model}$$