LECTURE #11

STAT MECH CORE

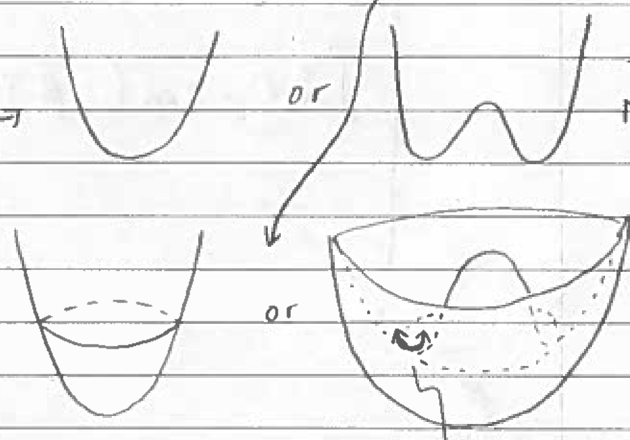
①

October 26, 2018

Summary:

$$Z = \int \underbrace{\prod dx_n^{(i)}}_{\equiv \vec{x}_N \text{ with } N \text{ components}} \exp\left[-\beta \sum_{\langle ij \rangle}{}' (\vec{x}_i - \vec{x}_j)^2 + \underbrace{m^2 \sum_i{}' \vec{x}^{\,2} + \mu^2 \sum_i{}' (\vec{x}^{\,2})^2}\right]$$

$$Z = \int \mathcal{D}\vec{S}_n \exp\left(-\beta J \sum_{ij}{}' \vec{S}_i \cdot \vec{S}_j\right)$$



or

for $N=1$

Mermin-Wagner

or

for $N \geq 2$

$\underbrace{N-1}_{\#\text{ broken symmetries}}$ Goldstone modes

$$\xrightarrow[\text{distances}]{\text{large}} \int d^D \vec{n}\, (\nabla \vec{x})^2 + m_{eff}^2 \vec{x}^{\,2}$$

exponential decay if $m_{eff} \neq 0$ (boring/empty at large distances)

power law decay if $m_{eff} = 0$ (interesting at large distances)

requires tuning!

$$\langle SS \rangle \sim \frac{1}{|x|^{\eta}}$$

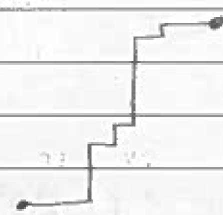
example of a critical exponent

$$Z = \int \mathcal{D}\underbrace{\vec{S}_{\vec{n}}}_{\text{unit spin vectors with N components}} \exp\left(-\beta J {\sum_{ij}}' \vec{S}_i \cdot \vec{S}_j\right)$$

$$= \begin{cases} \displaystyle {\sum_{S_i=\pm 1}}' \exp\left(-\beta J {\sum}' S_i S_j\right) & N=1 \text{ Ising model} \\ \displaystyle \int \mathcal{D}\theta_{\vec{n}} \exp\left(-\beta J {\sum}' \cos(\theta_i - \theta_j)\right) & N=2 \; XY \\ \vdots & \end{cases}$$

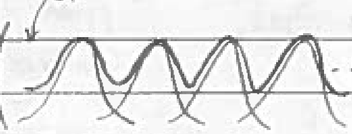
both are more general concepts! (not just XY)

XY Model

high temperature: $\langle SS \rangle \sim$  exp. decay

duality:

$$XY \simeq \int \prod d\theta_{\vec{n}} \prod_{\langle ij \rangle} \underbrace{\left(\text{Gaussians} \cdots\right)}_{}$$



$$= \sum_n e^{in\theta} \mathcal{F}(\wedge)(n)$$

can integrate out

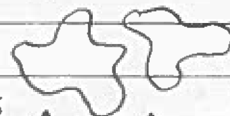
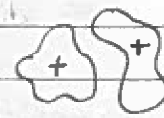
Fourier

$$\rightsquigarrow \sum_{m_a}' \prod_{\langle ab\rangle} \exp\left(\frac{-1}{2\beta J}(m_a - m_b)^2\right)$$ fluctuating surface model

↑ at each face

$$\beta \to \frac{1}{\beta} \quad \text{here!}$$

Aside: $\exists$ a high-T / low-T duality for Ising

$Z = \sum'_{\text{loops}}$  $\beta^{\text{length}} = \sum'_{\text{loops}}$  $e^{-\beta \cdot \text{length}}$

closed since $\sum \sigma_i = 0$ — high T — low T

Universality: same critical exponents

BUT

not same short-dist. behaviour,

not same T_c

↳ Example: we saw this in our table when adding next nearest neighbour interactions...

$\sin^2\frac{k}{2} \rightsquigarrow$ other function that $\simeq k^2$ at small k

$$\underset{\text{massless}}{\rightsquigarrow} \frac{1}{|x|^{D-2}}$$

From a field theory point of view:

$$\int d^D x \, (\nabla x)^2 + \underbrace{b\,(\nabla^2 x)^2 + \ldots}_{\text{higher derivatives}} \sim \int d^D k \left[x_k \big(k^2 + \underbrace{b^2 k^4 + \ldots}_{\text{irrelevant at small } k}\big) x_{-k} \right]$$

irrelevant operators

$\leftrightarrow$ large dist.

Or, from dimensional analysis:

$$\int \underbrace{d^D x}_{\dim = L^D} \underbrace{(\nabla x)^2}_{\dim = L^{-D}} + b(\nabla^2 x)^2 + \dots$$

∇ has dim $\frac{1}{L}$

$\Rightarrow \dim[x] = L^{-\left(\frac{D-2}{2}\right)}$

Then $\dim[(\nabla^2 x)^2] = L^{-D-2}$

$\Rightarrow \dim[b] = L^2$

$$\underset{\text{dimensionless}}{\text{Observable}(\vec{R})} = \mathcal{O}^{b=0} + \underbrace{\frac{b}{|\vec{R}|^{2\#}}}_{\substack{\to \infty \text{ at } R=0 \\ \to 0 \text{ at large dist.}}} \quad \text{by dim. analysis}$$

irrelevant operators do not move you to a new universality class

Real-space renormalization for the vortex gas:

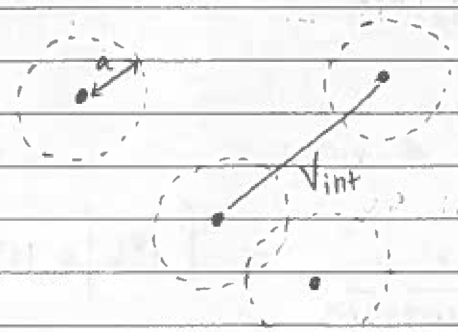
$$Z = \sum_{n} \frac{K^{2n}}{(n!)^2} \int_{|\vec{r}_i - \vec{r}_j| > a} \prod_{j=1}^{2n} d\vec{r}_j \, \exp\Big[\underbrace{2\pi\beta J \sum_{i\neq j} n_i n_j \log \frac{|\vec{r}_i - \vec{r}_j|}{a}}_{V_{int}}\Big]$$

n +1 vortices

n −1 vortices

$n_i = \pm 1$

cutoff

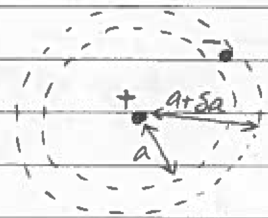


Goal: find Z for $a \to a + \delta a$

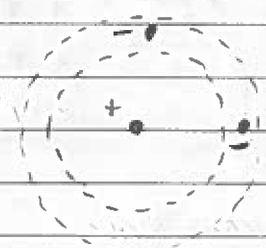
↳ want to integrate out $|\vec{r}_{ij}| \in [a, a+\delta a]$

$$Z = \cdots \int_{|\vec{r}_i - \vec{r}_j| > a + \delta a} \cdots$$

We need to be careful of cases where, under $a \to a+\delta a$, we can no longer see a pair of vortices:



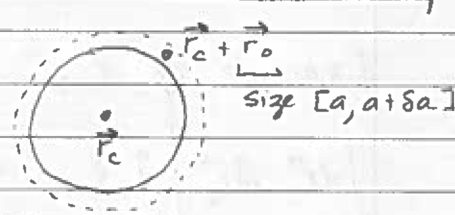
We will assume cases with ≥ 3 vortices are very unlikely and don't contribute (valid when the density of vortices is low enough; we will check this assumption later)



We get:

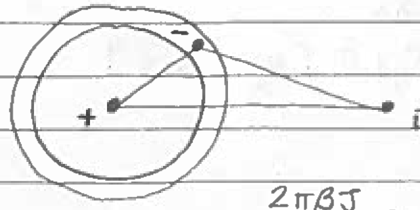
$$Z = \begin{array}{c}\text{Same expression}\\ \text{with cutoff}\\ a \to a+\delta a\end{array} + \underbrace{\begin{array}{c}\text{2 vortices}\\ \int d\vec{r}_c \, d\vec{r}_o\end{array}}_{\delta Z}$$

centre orbit



$$= Z\left[K(a) \to K^{eff} = K + \delta a X, \quad \beta J \to \beta J_{eff} = \beta J + \delta_a Y\right]$$

renormalization group equations



$$\frac{2\pi\beta J}{2}\, n_i \log \underbrace{\frac{|\vec{r}_i - \vec{r}_{centre}|^2}{|\vec{r}_i - \vec{r}_{centre} - \vec{r}_{orbit}|^2}}$$

$$\left[1 + \underbrace{2\,\frac{\vec{r}_{orbit}\cdot(\vec{r}_c - \vec{r}_i)}{|\vec{r}_c - \vec{r}_i|^2} + \frac{\vec{r}_{orbit}^{\,2}}{|\vec{r}_c - \vec{r}_i|^2}}_{\text{decays as } |\vec{r}_c - \vec{r}_i| \to \infty}\right]^{-1}$$

$$= 1 + \dots$$

$$\delta Z = \frac{K^{2n}}{(n!)^2} \underbrace{n^2}_{\substack{\text{choosing}\\ \text{positions}\\ 1\,\&\,2}} \int \underbrace{dr_c}_{r_1}\, \underbrace{dr_o}_{r_2} \underbrace{\prod_{k=3}^{n}\left(1 + \vec{r}_{orbit}\cdot(\dots) + \vec{r}_{orbit}^{\,2}(\dots)\right)}$$

$$1 + \underbrace{\text{linear in } \vec{r}_{orbit}}_{\text{vanishes since } \int d\vec{r}_{orbit}\, \vec{r}_{orbit}\cdot\vec{A} = 0} + \#\vec{r}_{orbit}^{\,2} + \#(\vec{r}_{orbit}\cdot\vec{A})(\vec{r}_{orbit}\cdot\vec{B})$$

Example of $\vec{r}_{orbit}^{\,2}$ term:

$$\int d\vec{r}_c\, d\vec{r}_o \left[\underline{1}_{\,*} + \dots + \sum_{\ell \neq k} \frac{16\pi^2\beta^2 J^2 n_k n_\ell}{|\vec{r}_c - \vec{r}_k|^2 |\vec{r}_c - \vec{r}_\ell|^2} \underbrace{(\vec{r}_c - \vec{r}_k)\cdot\vec{r}_o\ (\vec{r}_c - \vec{r}_\ell)\cdot\vec{r}_o}\right]$$

$$\int dr_o (r_o\cdot A)(r_o\cdot B) = A_a B_b \int dr_o\, r_a r_b$$

$$= 2\pi a^3 \delta a\, \vec{A}\cdot\vec{B}$$

$$\int d\vec{r}_o \rightsquigarrow 2\log|r_\ell - r_k| \ !$$

$$Z \simeq \underbrace{e^{\delta F}}_{\substack{\text{norm. from} \\ 1 \, (*) \text{ in} \\ \text{vortex pair} \\ \text{computation}}} \sum_{n=0}^{\infty}{}' \frac{K^{2n}}{(n!)^2} \int_{|\vec{r}_i - \vec{r}_j| > a + \delta a} d^2\vec{r}_i \; \exp\left[-\sum_{\langle ij \rangle}{}' \underbrace{\left(2\pi\beta J - 16\pi^3 \beta^2 J^2 K^2 a^3 \delta a \right)}_{(2\pi\beta J)'} n_i n_j \log \frac{|\vec{r}_{ij}|}{a} \right]$$

$\delta F \simeq (\text{Area}) \delta a$

Now $a_{new} = a + \delta a$

Once we incorporate this rescaling into the $\log \frac{|\vec{r}_{ij}|}{a}$, we will see that we get an RG equation for K.