Shape dependence of renormalized holographic entanglement entropy and Willmore energy

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Introduction

The Ryu-Takayanagi (RT) formula relates holographically the Entanglement Entropy (EE) of an entangling subregion in a Conformal Field Theory (CFT) with the area of a codimension-2 hypersurface immersed in Einstein-anti-de Sitter (AdS) spacetime. EE is obtained at the limit of Rényi entropy when the replica parameter m tends to 1. In the gravity side [1], this corresponds to a squashed-cone $(d+1)$-dimensional replica orbifold $M_{d+1}^{(\alpha)}$ whose angular deficit is $2\pi(1-\alpha)$ and $\alpha = 1/m$. Based on these considerations, the EE is defined as

$$S = -\lim_{\alpha\to 1} \partial_\alpha I_{\mathsf{E}}\left[\mathcal{M}_{d+1}^{(\alpha)}\right], \tag{1}$$

where $I_{\mathsf{E}}\left[\mathcal{M}_{d+1}^{(\alpha)}\right]$ is the Euclidean action evaluated on the orbifold $\mathcal{M}_{d+1}^{(\alpha)}$.

Renormalizing EE using Kounterterms

In the Kounterterms method [2], for even-dimensional manifolds $\mathcal{M}_{2n}$ with $2n = d+1$, the renormalized Einstein-AdS action is given by

$$I_{\mathsf{E}}^{\mathsf{ren}} = \frac{1}{16\pi G_N}\int\limits_{\mathcal{M}_{2n}} \mathrm{d}^{2n}x\sqrt{|G|}L^{2n-2}P_{2n}(\mathcal{F}) - \frac{(-1)^n}{4G_N}\frac{\pi^{(2n-1)/2}L^{2n-2}}{\Gamma[(2n-1)/2]}\chi\left[\mathcal{M}_{2n}\right],$$

where $\chi\left[\mathcal{M}_{2n}\right]$ is the Euler characteristic of $\mathcal{M}_{2n}$ and the polynomial of the AdS curvature $\mathcal{F}_{\nu_1\nu_2}^{\mu_1\mu_2} = R_{\nu_1\nu_2}^{\mu_1\mu_2} + \frac{1}{L^2}\delta_{\nu_1\nu_2}^{\mu_1\mu_2}$ reads

$$P_{2n}(\mathcal{F}) = \frac{1}{2^n n\Gamma(2n-1)}\sum_{k=1}^{n}\frac{(-1)^k[2(n-k)]!2^{(n-k)}}{L^{2n-k}}\binom{n}{k}\delta_{\mu_1\ldots\mu_{2k}}^{\nu_1\ldots\nu_{2k}}\mathcal{F}_{\nu_1\nu_2}^{\mu_1\mu_2}\cdots\mathcal{F}_{\nu_{2n-1}\nu_{2n}}^{\mu_{2n-1}\mu_{2n}}.$$

When $I_{\mathsf{E}}^{\mathsf{ren}}$ is evaluated in the orbifold, the geometrical quantities decompose into a regular part $\mathcal{M}_{2n}^{(\alpha)}\setminus\Sigma$ and another localized in the conical singularity Σ. The singular part reads

$$I_{\mathsf{E}}^{\mathsf{ren}}[\Sigma] = T\underbrace{\left(-\frac{L^{2n-2}}{2(2n-3)}\int\limits_{\Sigma}\mathrm{d}^{2n-2}y\sqrt{\gamma}P_{2n-2}(\mathcal{F}) - (-1)^n\frac{\pi^{(2n-1)/2}L^{2n-2}}{\Gamma[(2n-1)/2]}\chi[\Sigma]\right)}_{\mathsf{Vol}^{\mathsf{ren}}(\Sigma)}$$

This is the Nambu-Goto action of a cosmic brane with tension $T = \frac{1-\alpha}{4G_N}$ and renormalized volume $\mathsf{Vol}^{\mathsf{ren}}(\Sigma)$. Using (1), we find the renormalized version of RT formula $S^{\mathsf{ren}} = \frac{\mathsf{Vol}^{\mathsf{ren}}(\Sigma_{\mathsf{RT}})}{4G_N}$.
We identify two kinds of contributions to EE in,

$$S^{\mathsf{ren}} = -\frac{L^{d-1}}{8G_N(d-2)}\int\limits_{\Sigma_{\mathsf{RT}}}\mathrm{d}^{d-1}y\sqrt{\gamma}P_{d-1}(\mathcal{F}) + \frac{(-1)^{(d-1)/2}}{4G_N}\frac{\pi^{d/2}L^{d-1}}{\Gamma(d/2)}\chi\left[\Sigma_{\mathsf{RT}}\right], \tag{2}$$

- A curvature (local) term, depending on local integrals over Σ_{RT}.
- A topological (global) term, given $\chi\left[\Sigma_{\mathsf{RT}}\right]$.

When the entangling region is a perfect disk, the local term vanishes and the remaining contribution corresponds to the free energy of the CFT on $\mathbb{S}^d$.

Renormalized EE of a deformed disk

Consider a circular entangling region $\mathbb{S}^1$ in CFT_3. Contributions to EE [3] coming from the deformation of the entangling surface $\mathbb{S}_\epsilon^1$ are **universal** and they adopt the expansion

$$S(\mathbb{S}_\epsilon^1) = S^{(0)}(\mathbb{S}^1) + \epsilon^2 S^{(2)}(\mathbb{S}^1) + \mathcal{O}(\epsilon^3).$$

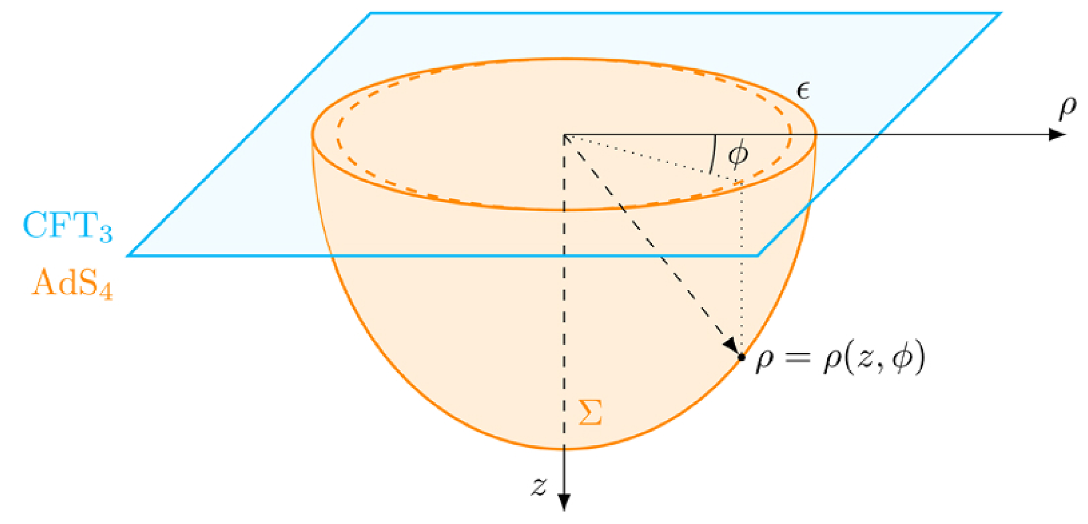
The leading correction is controlled by the **two-point function charge** $S_{\mathsf{ren}}^{(2)}(\mathbb{S}_\epsilon^1) = \frac{\pi^4 C_T}{24}\sum_\ell \ell(\ell^2-1)\left(a_\ell^2 + b_\ell^2\right)$.

In order to show this result, we start from the gravity side with the Poincaré-AdS metric

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\left(-\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}\rho^2 + \rho^2\mathrm{d}\phi^2\right)$$

and we parametrize the deformed minimal codimension-2 surface with the embedding function

$$\rho(z,\phi) = \sqrt{1-z^2}\left[1 + \epsilon\sum_\ell\left(\frac{1-z}{1+z}\right)^{\ell/2}\frac{1+\ell z}{1-z^2}\left(a_\ell\cos(\ell\phi) + b_\ell\sin(\ell\phi)\right)\right].$$



Then, using expression (2) in the case $d = 3$, given by

$$S^{\mathsf{ren}} = -\frac{\pi L^2}{2G_N}\chi(\Sigma_{\mathsf{RT}}) + \frac{L^2}{8G_N}\int_{\Sigma_{\mathsf{RT}}}\mathrm{d}^2x\sqrt{\gamma}\mathcal{F},$$

we match the result of [3]. The deformation contribution is encoded in the local term.

Willmore energy

Willmore energy measures the deviation of a surface X from the round sphere. It is defined by $\mathcal{W}(X) = \int\limits_X H^2\mathrm{d}s$, where H is the mean curvature. It acquires a **minimum value** when evaluated on a spherical submanifold $\mathcal{W}(X) \geq 4\pi$.
In [4], it was shown finite term of EE is the Willmore energy of the double copied RT surface in $\mathbb{R}^3$. Using formula (2), we relate the Willmore energy to renormalized EE,

$$S^{\mathsf{ren}} = -\frac{L^2}{8G_N}\mathcal{W}\left(2\Sigma_{\mathsf{RT}}\right).$$

In order to obtain this expression, we started from the renormalized volume of codimension-2 surface Σ and imposed the minimality condition. This gives an upper bound on deformations in the entangling surface

$$\int\limits_{\Sigma_{\mathsf{RT}}}\mathrm{d}^2y\sqrt{\gamma}\mathcal{F} \leq 0.$$

Conclusion

- Formula (2) identifies local and global contributions to EE for odd d.
- The free energy is found to have a origin coming from the topology of the RT surface
- The upper bound from Willmore energy gives a geometrical view of **strong subbaditivity**.
- When saturated, the local term gives no contribution and corresponds to the free energy.

References

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