



SAPIENZA  
UNIVERSITÀ DI ROMA



## SD@Convergence Workshop

# Gravitational Collapse in SD

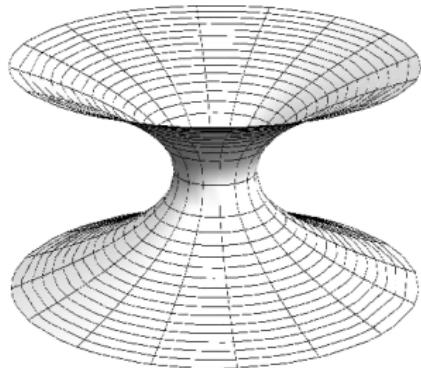
Andrea Napoletano  
Sapienza Università di Roma

In collaboration with:

Flavio Mercati  
Henrique Gomes  
Tim Koslowski

# Isotropic Wormhole Solution

$$ds^2 = \left( \frac{1 - \frac{\alpha}{4r}}{1 + \frac{\alpha}{4r}} \right)^2 dt^2 + \left( 1 + \frac{\alpha}{4r} \right)^4 (dr^2 + r^2 d\Omega_2)$$



- ▶ No Singularity
- ▶ Symmetry under inversion  $r \rightarrow \frac{\alpha^2}{16r}$
- ▶ Two Schwarzschild exteriors glued together at the throat

THE ISOTROPIC SOLUTION IS ETERNAL

*"A Birkhoff theorem for Shape Dynamics"* H. Gomes arXiv:1305.0310

# The Thin Shell Model

WHAT?

Dynamical creation of the isotropic solution through  
gravitational collapse

HOW?

Spherically symmetric infinitely thin shell of dust

# The Thin Shell Model: Starting Point

Metric tensor and conjugate momentum

$$g_{ij} = \begin{pmatrix} \mu^2 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \sin(\theta)^2 \end{pmatrix} \quad p^{ij} = \sin(\theta) \begin{pmatrix} \frac{f}{\mu} & 0 & 0 \\ 0 & \frac{1}{2}s & 0 \\ 0 & 0 & \frac{1}{2}s \sin(\theta)^{-2} \end{pmatrix}$$

Hamiltonian, diffeo and conformal constraints

$$\frac{1}{2\sigma\mu^2} \left[ 2f\sigma\mu^2s - f^2\mu^3 + \mu(\sigma')^2 + 4\sigma\mu^3 + 4\sigma\mu'\sigma' - 4\sigma\mu\sigma'' \right] = \delta(r - R) \sqrt{\frac{P^2}{\mu^2} + M^2}$$

$$\mu f' - \frac{1}{2}s\sigma' = -\frac{P}{2}\delta(r - R)$$

$$\mu f + s\sigma = 0$$

# Solution to Constraints

## Solutions

- ▶  $f = p^{rr}\mu = \frac{A}{\sqrt{\sigma}}$
- ▶  $\mu^2 = g_{rr} = \frac{(\sigma')^2}{4\alpha\sqrt{\sigma} + 4\sigma + \frac{A^2}{\sigma}}$

## Integration Constants

- ▶  $A_{\text{in}}, A_{\text{out}}$
- ▶  $\alpha_{\text{in}}, \alpha_{\text{out}}$

4 spatial integration constants  
2 inside the shell, 2 outside the shell

# Metric EoM and LFE

## Metric Tensor

$$\dot{g}_{ij} = \left( \frac{2f\mu^2 N}{\sigma} + 2\mu\xi\mu' + 2\mu^2\xi' \right) \delta^r{}_i \delta^r{}_j + \left( \frac{\sigma s N}{\mu} + \xi\sigma' \right) \left( \delta^\theta{}_i \delta^\theta{}_j + \delta^\phi{}_i \delta^\phi{}_j \sin^2 \theta \right)$$

## Conjugate Momentum

$$\begin{aligned} \dot{p}^{ij} = & - \frac{\sin \theta}{4\sigma\mu^3} \left[ 5f^2 N\mu^2 + 4\sigma(N'\sigma' - \mu^2\xi f') + N(-4\sigma\mu^2 + (\sigma')^2) + 4f\sigma\mu(\xi\mu' + \mu\xi') \right] \delta^i{}_r \delta^j{}_r \\ & + \frac{\sin \theta}{4\sigma^2\mu^2} \left[ (f^2 N\mu^3 + N\mu(\sigma')^2 + 2\sigma^2(2N'\mu' + \mu^2(\xi s' + s\xi') - 2\mu N'') \right. \\ & \left. - 2\sigma(-N\sigma'\mu' + \mu(N'\sigma' + N\sigma''))) \right] \left( \delta^i{}_\theta \delta^j{}_\theta + \delta^i{}_\phi \delta^j{}_\phi \sin^{-2} \theta \right) \\ & + \sin \theta \delta^i{}_r \delta^j{}_r \frac{P^2/\mu^4}{2\sqrt{P^2/\mu^2 + M^2}} N(R) \delta(r - R) \end{aligned}$$

## Lapse Fixing Equation

$$\begin{aligned} \frac{1}{4\sigma\mu^2} \left[ 6f^2 N\mu^3 + 4\sigma \left( -\mu \left( 2N'\sigma' + N\sigma'' \right) + N\sigma'\mu' + N\mu^3 \right) + N\mu(\sigma')^2 \right. \\ \left. + \sigma^2 \left( \mu \left( 3Ns^2 - 8N'' \right) + 8N'\mu' \right) \right] + \frac{P^2/\mu^2}{2\sqrt{P^2/\mu^2 + M^2}} N \delta(r - R) = 0 \end{aligned}$$

# Solutions to EoM

Lapse function

$$N = \frac{\sigma'}{2\mu\sqrt{\sigma}} \left( \textcolor{red}{c_1} + \textcolor{red}{c_2} \int_{r_a}^r dy \frac{\mu^3(y)}{(\sigma'(y))^2} \right)$$

Shift vector

$$\xi = \frac{\dot{\sigma}}{\sigma'} + \frac{\textcolor{red}{A}}{\sigma^{\frac{1}{2}}\sigma'} N(r)$$

Metric tensor and conjugate momentum

$$\dot{\alpha} = 0 \quad \textcolor{red}{c}_2 = -2\dot{A}$$

# Assumptions and jump conditions

## Integration Constants

$$A_{\text{in}} \ A_{\text{out}} \ \alpha_{\text{in}} \ \alpha_{\text{out}} \ c_{1\text{in}} \ c_{1\text{out}}$$

### Assumptions

- ▶ Continuity of the metric tensor
- ▶ No mass inside the shell
- ▶ Asymptotic flatness at infinity
- ▶ Continuity of the Lapse
- ▶  $N = 1$  at infinity

### Jump Conditions

- ▶ Diffeo constraint
- ▶ Hamiltonian constraint
- ▶ LFE
- ▶ EoM for  $p_{ij}$

$$A_{\text{in}} \ A_{\text{out}} \ \alpha_{\text{out}} \ c_{1\text{out}} \ \alpha_{\text{in}} = 0$$

# Reduced Phase Space

## On-Shell Relation

$$(\alpha_{\text{out}} + \rho) A_{\text{in}}^2 + \rho A_{\text{out}}^2 + \left( \frac{M^2}{16\rho} - \alpha_{\text{out}} - 2\rho \right) A_{\text{in}} A_{\text{out}} - \rho^3 (\alpha_{\text{out}})^2 - \frac{M^2 \rho^2}{8} \left( \frac{M^2}{32\rho} - \alpha_{\text{out}} - 2\rho \right) = 0$$

## Independent Variables

- $\rho = \sigma(R) = g_{\theta\theta}(R)$  Area of the Shell
- $A_{\text{in}}$   $A_{\text{out}}$  Related to the Momentum  $P$
- $\alpha_{\text{out}}$  Related to the ADM mass

## UNDERDETERMINED SYSTEM

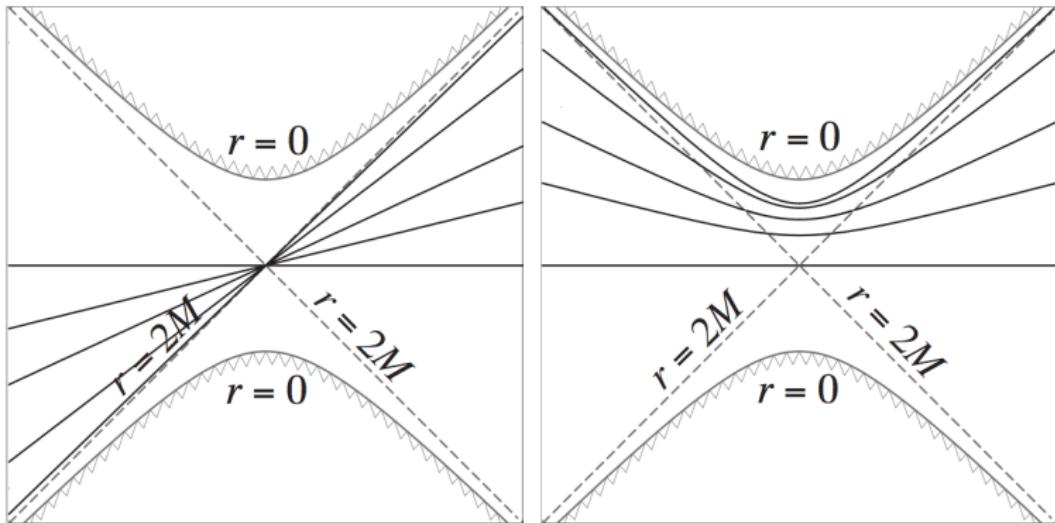
The on-shell relation and the condition  $\alpha_{\text{out}} = \text{const.}$  define a 2-dimensional manifold - not a one dimensional curve

# What is the under determination?

- ▶ Spherically symmetric thin shell
- ▶ Conformal constraint  $g_{ij}p^{ij} = 0$
- ▶ General result: Birkhoff Theorem

Schwarzschild space-time in maximal foliation

MAXIMAL FOLIATION OF SCHWARZCHILD IS NOT UNIQUE



# GR point of view

Isotropic static foliation  $A_{\text{out}} = 0$

$$ds^2 = \left( \frac{1 - \frac{\alpha}{4r}}{1 + \frac{\alpha}{4r}} \right)^2 dt^2 + \left( 1 + \frac{\alpha}{4r} \right)^4 (dr^2 + r^2 d\Omega_2)$$

C foliation  $C \leftrightarrow A_{\text{out}}$

$$ds^2 = f(C(\tau)) \dot{C}(\tau)^2 d\tau^2 + g(C(\tau)) \dot{C}(\tau) d\tau dr + \frac{1}{1 + \frac{\alpha}{y} + \frac{C(\tau)^2}{4y^2}} dy^2 + y^2 d\Omega_2^2.$$

A 4-dimensional diffeomorphism connects the two solutions  
THERE IS NO AMBIGUITY IN GR

# Why then the ambiguity in SD?

- ▶ SD rests on a preferred notion of simultaneity
- ▶ SD symmetries are 3d diffeomorphisms and 3d conformal transformations
- ▶ The 2 maximal foliations of Schwarzschild are not connected by a 3d conformodiffeo

What fixes  $A_{\text{out}}$ ?

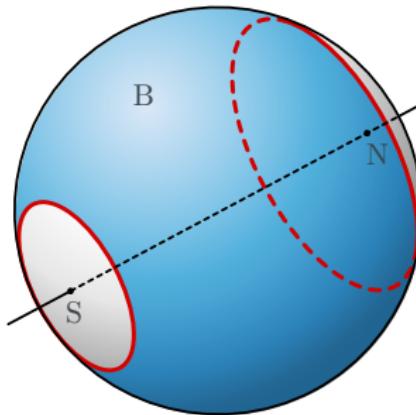
The rest of the universe

# Twin Shell Model

SD is naturally defined in a compact universe

A spherically symmetric compact universe with  $S^3$  topology is the simplest model

$$ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)$$



It requires at least two thin shell to not have any mass at the origin

# The twin shell model: Starting point

Ansatz

$$g_{ij} = \text{diag} \left\{ \mu^2, \sigma, \sigma \sin^2 \theta \right\} \quad p^{ij} = \text{diag} \left\{ \frac{f}{\mu}, \frac{1}{2}s, \frac{1}{2}s \sin^{-2} \theta \right\} \sin \theta \quad \xi^i = \{\xi, 0, 0\}$$

Hamiltonian Constraint

$$\int d\theta d\phi \left[ \frac{p_T^{ij} p_{ij}^T}{\sqrt{g}} - \frac{1}{6} \left( \langle p \rangle^2 - 12\Lambda \right) \sqrt{g} - \sqrt{g} R \right] + 4\pi \sum_{a \in \{\mathbf{S}, \mathbf{N}\}} \delta(\psi - \Psi_a) \sqrt{g^{rr} P_a^2 + M_a^2} = 0$$

Diffeo constraint

$$-2 \int d\theta d\phi \nabla_j p^j_i - 4\pi \delta^\psi_i \sum_{a \in \{\mathbf{S}, \mathbf{N}\}} \delta(\psi - \Psi_a) P_a = 0$$

Conformal constraint: CMC Foliation

$$g_{ij} p^{ij} - \sqrt{g} \langle p \rangle = 0$$

# Solutions to EoM and OS Relation

Diffeo and Hamiltonian constraints

$$f = \frac{1}{3} \langle p \rangle \sigma + \frac{A}{\sigma^{\frac{1}{2}}} \quad \mu^2 = \frac{(\sigma')^2}{\left( \frac{2}{3} A \langle p \rangle + 4\alpha \right) \sigma^{\frac{1}{2}} + 4\sigma + \frac{A^2}{\sigma} + \frac{1}{9} (\langle p \rangle^2 - 12\Lambda) \sigma^2}$$

We get two on shell relations

$$\frac{M_S^4}{16\rho_S^2} - M_S^2 \left( \frac{A_B A_S}{\rho_S^4} + 4 - T_{\rho_S}^2 \right) - 4 (A_B - A_S)^2 \left( \frac{16}{\rho_S^2} - 4T \right)$$

$$- \left( \frac{4}{\rho_S^3} A_S (A_S - A_B) + \frac{M_S^2}{2\rho_S} \right) X + X^2 = 0$$

$$\frac{M_N^4}{16\rho_N^2} - M_N^2 \left( \frac{A_B A_N}{\rho_N^4} + 4 - T_{\rho_N}^2 \right) - 4 (A_B - A_N)^2 \left( \frac{16}{\rho_N^2} - 4T \right)$$

$$- \left( \frac{4}{\rho_N^3} A_N (A_N - A_B) + \frac{M_N^2}{2\rho_N} \right) X + X^2 = 0$$

$$\rho^2 = \sigma(\Psi) \quad X = \left( \frac{2}{3} A_B \langle p \rangle + 4\alpha_B \right) \quad T = \frac{1}{9} (12\Lambda - \langle p \rangle^2)$$

# Twin Shell DoF

- ▶  $\rho_S \rho_N$
- ▶  $A_S A_N A_B$
- ▶  $\alpha_S \alpha_N \alpha_B$
- ▶  $\langle p \rangle V$
- ▶ 2 on shell relations
- ▶  $\alpha_S = 0 \quad \alpha_N = 0; \alpha_B = const$

4 matter DoF (2 for each shell)  
2 scale DoF (volume and momentum)

THE SYSTEM IS FULLY DETERMINED

# Towards Asymptotic Flatness

- ▶ Take the limit  $A_N \rightarrow \infty$
- ▶ Set to 0 each of the coefficients of  $A_N$
- ▶ 3 equations that can be solved in terms of  $\rho_N$   $\alpha_B$   $A_B$

2 possible solutions

$$\rho_N = \sqrt{\frac{4 - \sqrt{16 - M_N^2 T}}{2T}}, \quad \alpha_B = -\sqrt{\frac{4 - \sqrt{16 - M_N^2 T}}{2T}} \left( \frac{1}{8} \sqrt{16 - M_N^2 T} + \frac{1}{2} \right), \quad A_B = 0$$
$$\rho_N = \sqrt{\frac{\sqrt{16 - M_N^2 T} + 4}{2T}}, \quad \alpha_B = \left( \frac{1}{8} \sqrt{16 - M_N^2 T} - \frac{1}{2} \right) \sqrt{\frac{\sqrt{16 - M_N^2 T} + 4}{2T}} \quad A_B = 0$$

Asymptotic flatness becomes a good approximation for  $T \rightarrow 0$

$$V \propto \frac{1}{T} \Leftrightarrow \lim_{T \rightarrow 0} V = \infty$$

# Deriving $A_{\text{out}} = 0$

Late time limit  $T \rightarrow 0$

$$\rho_N \rightarrow \frac{M_N}{4}$$

$$\alpha_B \rightarrow -\frac{M_N}{4}$$

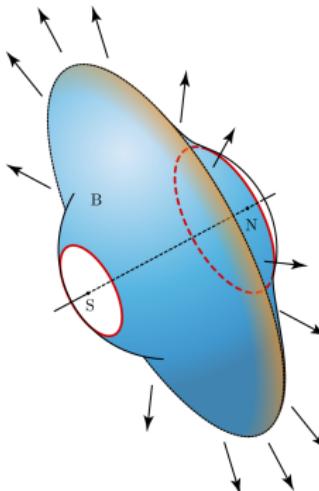
$$A_B \rightarrow 0$$

$$\rho_N \sim \frac{2}{\sqrt{T}} \rightarrow \infty$$

$$\alpha_B \sim -\frac{M_N^2}{32} \sqrt{T} \rightarrow 0^-$$

$$A_B \rightarrow 0$$

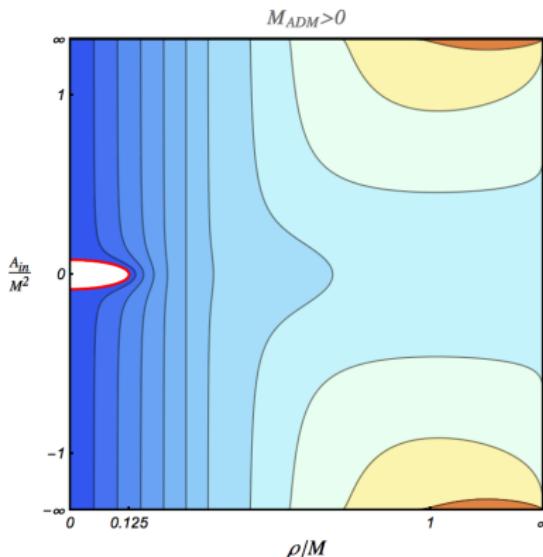
$$\boxed{\rho_N \rightarrow \frac{M_N}{4} \quad A_{\text{out}} = A_B = 0 \quad \alpha_{\text{out}} = \alpha_B = -\frac{M_N}{4}}$$



# Reduced phase space $A_{\text{out}} = 0$

On Shell Relation for the Thin Shell

$$(\alpha_{\text{out}} + \rho) A_{\text{in}}^2 - \rho^3 (\alpha_{\text{out}})^2 - \frac{M^2 \rho^2}{8} \left( \frac{M^2}{32\rho} - \alpha_{\text{out}} - 2\rho \right) = 0$$



$A_{\text{out}}$  diverges when the dynamics freezes or the shell escapes

# Thin Shell Symplectic Reduction

Symplectic potential

$$\begin{aligned}\theta = & \int dr d\theta d\phi p^{ij} \delta g_{ij} + 4\pi P \delta R \\ & - 8\pi \left( \delta A_{in} \int_0^R dr \frac{\mu}{\sqrt{\sigma}} + \delta A_{out} \int_R^\infty dr \frac{\mu}{\sqrt{\sigma}} \right) + 4\pi P \delta R\end{aligned}$$

Symplectic form

$$\omega = \delta \theta = -8\pi \left( \delta A_{in} \frac{\mu(R)}{\sqrt{\sigma(R)}} - \delta A_{out} \frac{\mu(R)}{\sqrt{\sigma(R)}} \right) \wedge \delta R,$$

$$\boxed{\omega = 4\pi \delta P \wedge \delta R}$$

$P$  and  $R$  are the conjugate dynamical variables

# ADM Mass

## Definition

$$M_{\text{ADM}} = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int d\theta d\phi \left( \bar{\nabla}^j g_{rj} - \bar{\nabla}_r \bar{g}^{ij} g_{ij} \right) r^2 \sin \theta$$

$$M_{\text{ADM}} = \frac{1}{2} \lim_{r \rightarrow \infty} \left( r \mu^2 - \sigma' + \frac{1}{r} \sigma \right)$$

Assumption: Asymptotic flatness  $\lim_{r \rightarrow \infty} \sigma = r^2$

$$M_{\text{ADM}} = -\frac{\alpha_{\text{out}}}{2}$$

$\alpha_{\text{out}}$  plays the role of the ADM Hamiltonian

# Thin Shell $A_{\text{out}} = 0$ : Isotropic Gauge

Isotropic gauge condition &  $A_{\text{out}} = 0$

$$\frac{\mu^2 r}{\sigma(r)} = \frac{1}{r^2} \quad \Rightarrow \quad \sigma(r) = r^2 \left(1 - \frac{\alpha_{\text{out}}}{4r}\right)^4$$

EoM in terms of  $R$  and  $P$ :  $\alpha_{\text{out}}(R, P)$

- ▶ On shell relation

$$(\alpha_{\text{out}} + p) A_{\text{in}}^2 - p^3 (\alpha_{\text{out}})^2 - \frac{M^2 p^2}{8} \left( \frac{M^2}{32p} - \alpha_{\text{out}} - 2p \right) = 0$$

- ▶ Isotropic relation

$$p^2 = R^2 \left(1 - \frac{\alpha_{\text{out}}}{4R}\right)^4$$

- ▶ Diffeo jump condition

$$A_{\text{in}} - \frac{RP}{2} = 0$$

The solution of this system  $\alpha_{\text{out}} = \alpha_{\text{out}}(R, P)$

# Thin Shell EoM

$$(\dots)d\alpha_{\text{out}} + (\dots)dA_{\text{in}} + (\dots)d\rho + (\dots)dR + (\dots)dP = 0$$

$$(\dots)d\alpha_{\text{out}} + (\dots)dA_{\text{in}} + (\dots)d\rho + (\dots)dR + (\dots)dP = 0$$

$$(\dots)d\alpha_{\text{out}} + (\dots)dA_{\text{in}} + (\dots)d\rho + (\dots)dR + (\dots)dP = 0$$

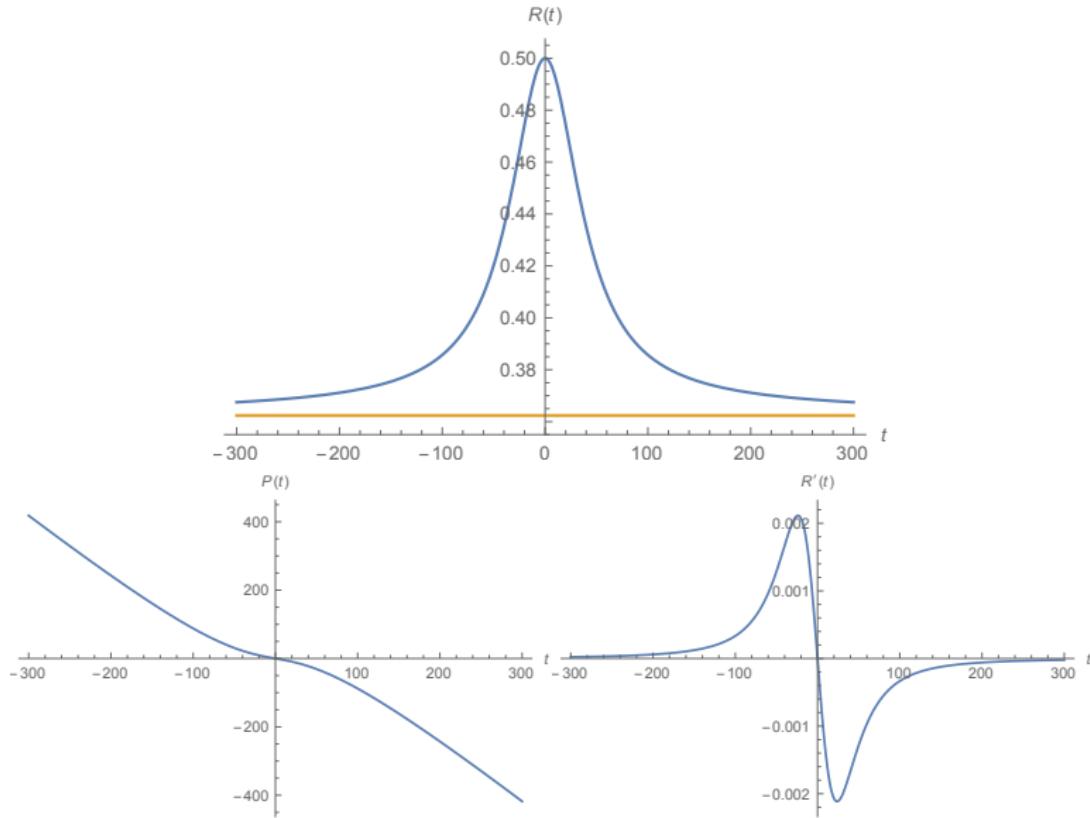
$$d\alpha_{\text{out}} = \left( \frac{\partial \alpha_{\text{out}}}{\partial R} \Big|_{A_{\text{in}} = \frac{RP}{2}, \rho = R \left(1 - \frac{\alpha_{\text{out}}}{4R}\right)^2} \right) dR + \left( \frac{\partial \alpha_{\text{out}}}{\partial P} \Big|_{A_{\text{in}} = \frac{RP}{2}, \rho = R \left(1 - \frac{\alpha_{\text{out}}}{4R}\right)^2} \right) dP$$

$$dA_{\text{in}} = \left( \frac{\partial A_{\text{in}}}{\partial R} \Big|_{A_{\text{in}} = \frac{RP}{2}, \rho = R \left(1 - \frac{\alpha_{\text{out}}}{4R}\right)^2} \right) dR + \left( \frac{\partial A_{\text{in}}}{\partial P} \Big|_{A_{\text{in}} = \frac{RP}{2}, \rho = R \left(1 - \frac{\alpha_{\text{out}}}{4R}\right)^2} \right) dP$$

$$d\rho = \left( \frac{\partial \rho}{\partial R} \Big|_{A_{\text{in}} = \frac{RP}{2}, \rho = R \left(1 - \frac{\alpha_{\text{out}}}{4R}\right)^2} \right) dR + \left( \frac{\partial \rho}{\partial P} \Big|_{A_{\text{in}} = \frac{RP}{2}, \rho = R \left(1 - \frac{\alpha_{\text{out}}}{4R}\right)^2} \right) dP$$

$$\dot{R} = -\frac{1}{2} \frac{d\alpha_{\text{out}}}{dP}(R, P, \alpha_{\text{out}}) \quad \dot{P} = +\frac{1}{2} \frac{d\alpha_{\text{out}}}{dR}(R, P, \alpha_{\text{out}})$$

# Thin Shell Numerical Simulations



# Conclusions

- ▶ The asymptotically flat picture is incomplete, SD deals properly only with closed compact universes.
- ▶ Asymptotic flatness can be understood only as an approximation to a void region into a compact universe. The matter in the rest of the universe sets the scale and defines a notion of simultaneity
- ▶ With the condition  $A_{\text{out}} = 0$ , the shell never reaches the horizon in maximal slice. This is not the end of the story because other physical inputs are currently being studied. Furthermore the dynamics of the twin shell is yet to be understood.