Lecture 2: Black holes & AdS

Recap of L1:

GR: "gravity" $\leftrightarrow$ ST curvature

- ST tells matter how to move   (eg. geodesic eqn for test particles)

$$\{x^\mu(\lambda)\} \quad \hookleftarrow \quad \ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta = 0$$

- matter tells ST how to curve   (Einstein's eqn)

$$\{g_{\mu\nu}(x), T_{\mu\nu}(x)\} \quad \hookleftarrow \quad R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu} + \Lambda\,g_{\mu\nu} = 8\pi G_N\, T_{\mu\nu}$$

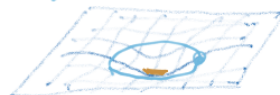Cf. Newtonian gravity, eg. Earth orbiting the Sun:

Newton: Earth accelarates   because it's acted on by force of gravity due to Sun's mass
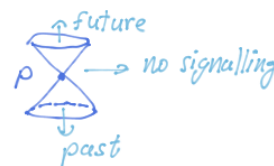
Einstein: Earth follows a geodesic in a curved ST   (curved by Sun's mass)

2-d analogy:

$\therefore$ ST in GR is curved & dynamical

w/ Lorentzian structure: $\exists$ light cone @ every point

↑ future

P → no signalling

↓ past

$\therefore$ metric $g_{\mu\nu}$ is not positive definite:

$$ds^2 = 0 \quad \text{for null (lightlike) curves}$$
$$ds^2 < 0 \quad \text{for timelike curves}$$

- E.eq. $\to$ hyperbolic system (once we know initial conditions on a spacelike "Cauchy" slice, we can evolve uniquely)

- $\{$Light cone $\forall p \in M\}$ $\to$ causal structure $\to$ determines where can signals propagate.

2A. Black holes:

- ### Schwarzschild soln:

  = static, spherically symmetric vacuum $\Lambda = 0$ soln. of E.eq.

  $$R_{\mu\nu} = 0 \qquad \leadsto \qquad g_{\mu\nu} = ?$$

  use coords adapted to symmetries $\qquad (t, r, \theta, \varphi)$

  static $\leadsto$ $g_{\mu\nu}$ indep. of t, $\quad g_{ti} = 0$

  spher. sym. $\leadsto$ (θ, φ) part: $\quad d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$

  choose r to measure proper area of S^2 5

  general ansatz: $\quad ds^2 = -f(r)\,dt^2 + h(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$

  $\rightarrow$ E.eq's are fairly simple:

  $$0 = R_{tt} = f \cdot \left\{ \tfrac{1}{2}(fh)^{-1/2}\tfrac{d}{dr}\left[(fh)^{-1/2} f'\right] + (rfh)^{-1} f' \right\}$$

  $$0 = R_{rr} = h \cdot \left\{ -\tfrac{1}{2}(fh)^{-1/2}\tfrac{d}{dr}\left[(fh)^{-1/2} f'\right] + (rh^2)^{-1} h' \right\}$$

  $$0 = R_{\theta\theta} = \tfrac{1}{\sin^2\theta} R_{\varphi\varphi} = r^2 \cdot \left\{ -\tfrac{1}{2}(rfh)^{-1} f' + \tfrac{1}{2}(rh^2)^{-1} h' + r^{-2}(1-h^{-1}) \right\}$$

  $\leadsto$ Soln: $\quad f(r) = \dfrac{1}{h(r)} = 1 - \dfrac{2M}{r}\dfrac{G_N}{c^2} \qquad$ (work in units s.t. $c = G_N = 1$)

  $M =$ mass by comparing w/ Newtonian gravity @ large r

  $\hookrightarrow \therefore$ we have a 1-parameter family of solns.

  (= most general vacuum asymptotically flat static, sph.sym. soln)

$M = 0 \leadsto$ Minkowski ST ($\Rightarrow$ flat)

$M > 0 \rightarrow$ describes ST outside spherical object (eg. star — inside which $T_{\mu\nu} \neq 0$)

           $\downarrow$

     + if $T_{\mu\nu} = 0$ $\forall$ $r > r_0$ w/ $r_0 < 2M$, corresponds to a black hole.

$M > 0$: $r \to \infty$, $ds^2 \to$ Minkowski : "asymptotically flat"

$r = 2M$: $g_{tt} \to 0$, $g_{rr} \to \infty$: coordinate singularity

  can be removed via coord. transformation

  $\rightsquigarrow$ "event horizon" (~ point of no return)

$r = 0$: $g_{tt} \to \infty$, $g_{rr} \to 0$: curvature singularity

$$R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{48 M^2}{r^6} \xrightarrow[r \to 0]{} \infty$$

  can be reached @ finite affine param. but ST breaks down!

Geometry probed by geodesics in $ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + \underbrace{r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)}_{\equiv d\Omega^2}$

symmetries $\twoheadrightarrow$ CoMs:

∂_t $\twoheadrightarrow$ $E = -\left(1 - \frac{2M}{r}\right) \dot{t}$ $\rightsquigarrow$ "energy" / unit mass

∂_φ $\twoheadrightarrow$ $L = r^2 \sin^2\theta \, \dot{\varphi}$ $\rightsquigarrow$ "angular momentum" / unit mass

  spher. sym. $\implies$ WLOG consider equatorial orbit $\rightsquigarrow$ set $\theta = \pi/2$

  norm of tangent vector $p_\mu p^\mu \equiv \kappa = \begin{cases} -1 & \text{timelike} \\ 0 & \text{null} \\ 1 & \text{spacelike} \end{cases}$ = const

Recast into 1-d motion in effective potential $\frac{1}{2} \dot{r}^2 + V_{eff}(r) = 0$

  ~ 'kinetic energy' 'pot. en.' tot. en.

w/ $V_{eff}(r) = -\frac{(E^2 + \kappa)}{2} + \kappa \frac{M}{r} + \frac{L^2}{2r^2} - \frac{L^2 M}{r^3}$

Physical interpretation:

  rest is usual ...
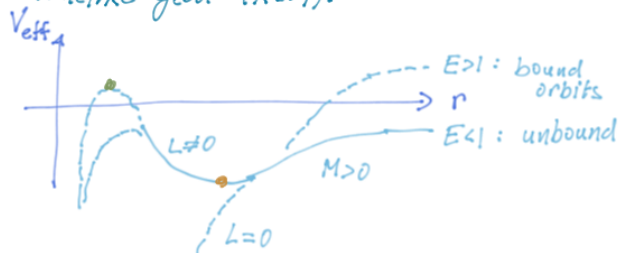
  New (GR effect)
  $\to$ angular momentum barrier
  $\to$ Newtonian potential
  $\to$ Rest mass contribution

eg. timelike geod ($\kappa = -1$):

  $E > 1$: bound orbits
  $E < 1$: unbound

$\bullet$ $\exists$ stable & unstable circular orbits
  $\to$ novel : GR effect

$\bullet$ precession of perihelion

null geods: - light bending   (→ gravitational lensing)
           • ∃ circular orbits   (→ "Einstein rings" around BH, ...)
           • gravitational redshift, grav. time delay
                     ⋮

• non-trival causal structure:

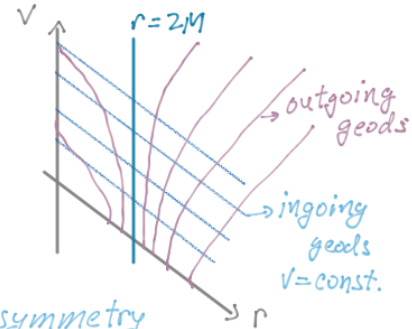   To get rid of coordinate singularity, choose coordinates adapted to ingoing null geods:

   let $v = t + r_*$ → tortoise coord, w/ $dr_* = \dfrac{dr}{(1 - \frac{2M}{r})}$

   → in Ingoing Eddington coords (v, r, θ, φ),

   $$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2 d\Omega^2$$

   ↳ regular @ $r = 2M$, but no longer $t \leftrightarrow -t$ symmetry

                  & $v = -\infty$ reached in finite affine par. ...

   → global extension of Schwarzschild: 1) (v, u, θ, φ) restores time-reversal sym.
                                              ↳ $t - r_*$

   2) Kruskal coords: (V, U, θ, φ): extend past $u, v = \pm\infty$: adapt coords to affine par. along null geods

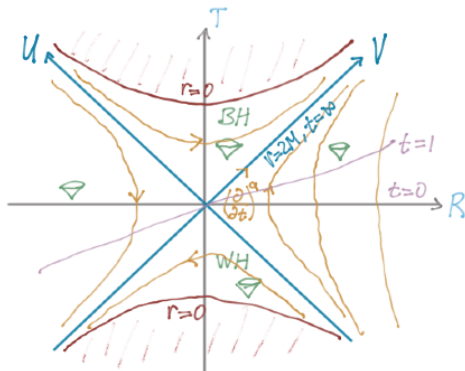 $$V = e^{v/4M} \quad, \quad U = -e^{-u/4M} \qquad \Rightarrow \quad du\, dv = -\frac{16M^2}{UV}\, dU\, dV$$

      w/ $UV = -e^{r_*/2M} = \left(1 - \frac{r}{2M}\right) e^{r/2M}$

      $$ds^2 = -\frac{32M^2}{r} e^{-r/2M}\, dU\, dV + r^2 d\Omega^2$$

            $\underbrace{\qquad\qquad}$
            regular @ $r = 2M$ ⇒ can extend $V \in (0, \infty) \to (-\infty, \infty)$
                                      & $U \in (-\infty, 0) \to (-\infty, \infty)$

   Kruskal diagram:

   Can also define $\begin{array}{l} V = T + R \\ U = T - R \end{array} \quad \Leftrightarrow \quad \begin{array}{l} T = \frac{V+U}{2} \\ R = \frac{V-U}{2} \end{array}$

   • $r \to 0$: $UV = T^2 - R^2 \to 1$ → limits ST region

   • $r = $ const $\leftrightarrow UV = T^2 - R^2 = $ const; = orbit of $\left(\frac{\partial}{\partial t}\right)^a$ K.V.

   • $t = $ const. $\leftrightarrow U/V = $ const ; diff. Im(t) in 4 regions

   • light cones @ 45° everywhere   (by construction)
              → read-off causal structure

Penrose diagram for Schwarzschild:

Compactify s.t. boundary is drawn @ finite distance, but light cones remain @ 45°
everywhere

⇝ causal structure manifest (but geometry obscure)
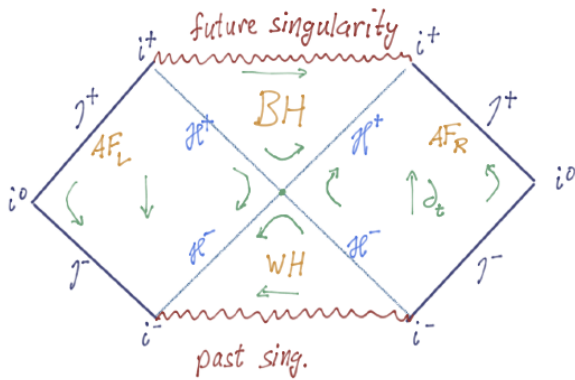
full ST has 4 regions
- $AF_{L,R}$ = left & right (causally disconnected from each other)
  asymp. flat regions
- BH = black hole region (can't see into from outside)
- WH = white hole (= time-reverse of BH)

- Separated by future & past event horizons $\mathcal{H}^{\pm}$ @ r=2M

- $\genfrac{}{}{0pt}{}{BH}{WH}$ contain $\genfrac{}{}{0pt}{}{future}{past}$ curvature singularity @ r=0   → terminates ST

- Const. r surfaces lie along orbits of ∂_t → $\begin{cases} \text{timelike for} & r>2M \\ \text{null for} & r>2M \\ \text{spacelike for} & r<2M \end{cases}$
  (zero @ bifurcation surface in middle)

- ST bdy @ ∞ affine param. along geods consists of
  - $i^{\pm}$ = $\genfrac{}{}{0pt}{}{future}{past}$ timelike infinity   (= "endpoint" of timelike geods)

  "scri" ⇝ $\mathcal{J}^{\pm}$ = — // — null — // —   — // — null — // —
  - i^0 = spatial infinity   (— // — spacelike — // —)

- each point on P.D. = S^2 w/ proper area $4\pi r^2$

- light cone @ each point locally ⧖ @ 45°

  eg. once inside BH, any observer must irrevocably fall into singularity...

More general BHs:

- **other soln's:**
  - add rotation (Kerr), charge (Reissner-Nordstrom), or both (Kerr-Newman accelerating BH (c-metric), multi BH (Majumdar-Papapetrou), ...
  - change asymptotics ($\Lambda > 0 \Rightarrow$ Schw-dS ; $\Lambda < 0 \Rightarrow$ Schw-AdS)
    ↳ see below
  - change dimensions
    - $\exists$ more exotic BH solns. in higher dim's — eg. black rings $\bigcirc$, ...
    ⋮

- **general defn.** (for ST w/ $\mathcal{I}^+$):

  event horizon $= \partial I^-(\mathcal{I}^+)$

  boundary of the past* of future null ∞

  BH = region inside event horizon = $M - I^-(\mathcal{I}^+)$
  $\Rightarrow$ region of no escape ...

  $\Rightarrow$ "teleological" : need to know the entire ST before we can determine the location of $\mathcal{H}^+$ (depends on what falls in later ...)

  N.B. $\exists$ a "quasilocal" notion of BH, eg. via "apparent horizon"
  ~ gravity locally prevents light from getting out ...

- **BH dynamics** (near-stationarity) ~ **thermodynamics** — cf. Focus Lecture

  eg. horizon area ~ entropy
  BH area thm: horizon area cannot decrease in any physical process
  & BHs have associated temperature

  ⤳ BHs are crucially important, eg. in holography (AdS/CFT)
  (dual to thermal states)

2B. AdS:

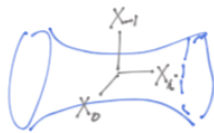= maximally symmetric soln. to vacuum $(T_{\mu\nu} = 0)$, $\Lambda < 0$ E.eq.

     (has const. $\ominus_{ve}$ curvature $\to$ Lorentzian analog of a hyperboloid)

- can obtain AdS_{d+1} by isometric embedding into $\mathbb{R}^{2,d}$

     as surface $-X_{-1}^2 - X_0^2 + X_1^2 + \cdots + X_d^2 = -\ell^2$ $\Rightarrow SO(d,2)$

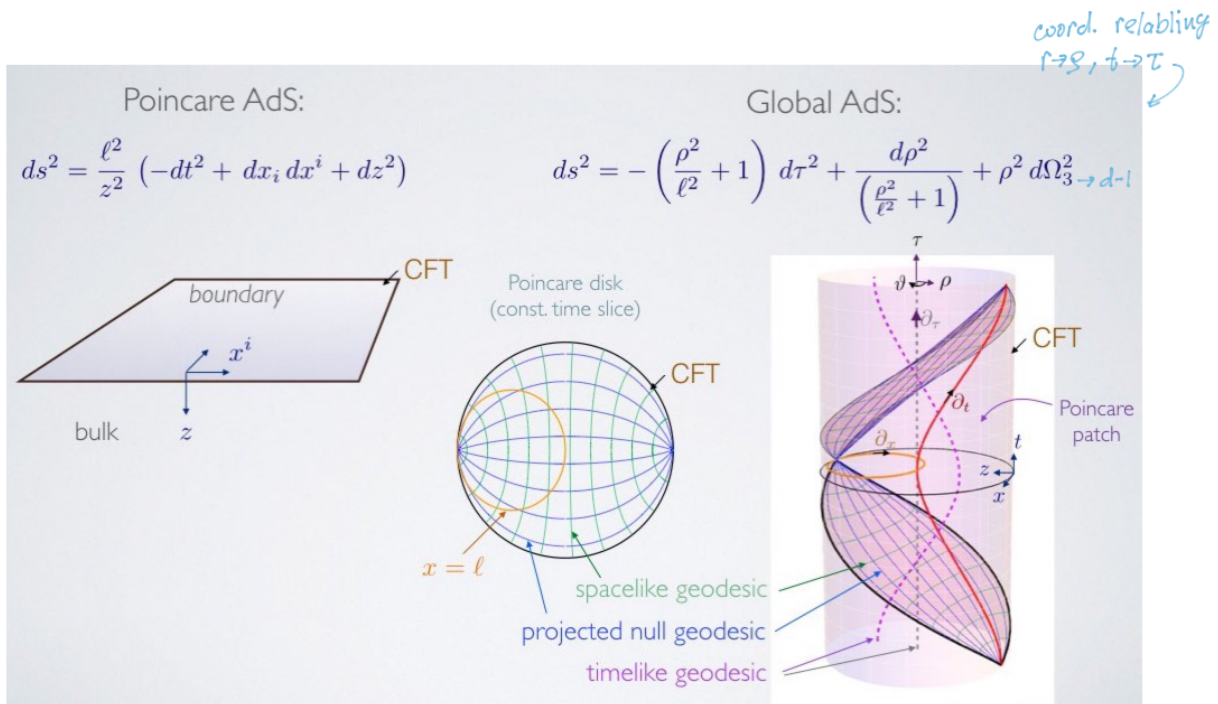   in $ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + \cdots + dX_d^2$

       $\cdot$ decompactify time

- in "global" coords: $ds^2 = -\left(\frac{r^2}{\ell^2} + 1\right) dt^2 + \frac{dr^2}{\left(\frac{r^2}{\ell^2} + 1\right)} + r^2 d\Omega_{d-1}^2$

     $r = \infty$: boundary $(\mathcal{I}^{\pm}, i^0)$ of AdS $\to$ timelike

- in "Poincaré" coords: $ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + dx_i dx^i + dz^2\right)$

             "boundary coords"   "radial coord"

coord. relabling
$r \to \rho, t \to \tau$

Poincaré AdS:
$$ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + dx_i dx^i + dz^2\right)$$

Global AdS:
$$ds^2 = -\left(\frac{\rho^2}{\ell^2} + 1\right) d\tau^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{\ell^2} + 1\right)} + \rho^2 d\Omega_{3 \to d-1}^2$$

CFT

boundary

x^i

bulk z

Poincaré disk (const. time slice)

CFT

$x = \ell$

spacelike geodesic

projected null geodesic

timelike geodesic

τ

$\vartheta \leftrightarrow \rho$

∂_τ

CFT

∂_t

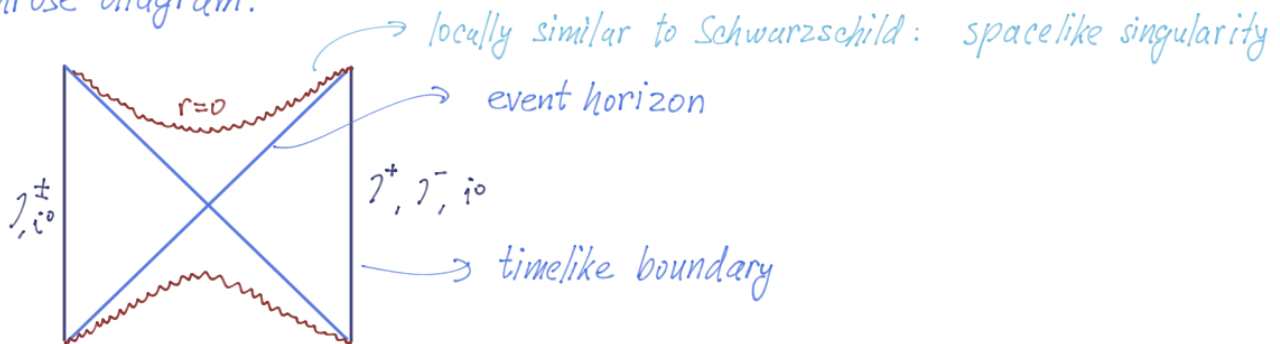Poincaré patch

∂_ρ

z

x

Schwarzschild – AdS:

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2 \qquad \text{in} \quad d+1 \quad \text{ST dimensions}$$

$$\text{w/} \quad f(r) = \frac{r^2}{\ell^2} + 1 - \underbrace{\left(\frac{r_0}{r}\right)^{d-2}}_{} \qquad \longrightarrow \text{alternately} \quad \left(\frac{r_+}{r}\right)^{d-2}\left(\frac{r_+^2}{\ell^2}+1\right)$$

$$\text{Mass of BH:} \quad M = \frac{(d-1)\,A_{d-1}\,r_0^{d-2}}{16\pi\,G_N} \qquad \longrightarrow \text{area of unit } S^{d-1}$$

- Penrose diagram:

→ locally similar to Schwarzschild: spacelike singularity

→ event horizon

$r=0$

$J^{\pm}_{,\,i^0}$ J^+, J^-, i^0

→ timelike boundary

- Temperature:

$$T = \frac{f'(r_+)}{4\pi} = \frac{d\,r_+^2 + (d-2)\,\ell^2}{4\pi\,r_+\,\ell^2}$$
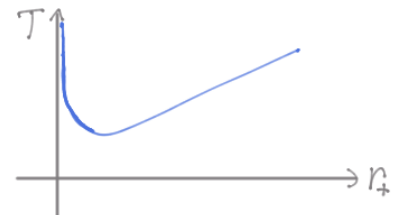
- small BHs are thermodynamically unstable (have $\ominus_{ve}$ specific heat)
- large BHs —— // —— stable $\oplus$ —//—

    (can be in equil. w/ Hawking radiation due to AdS attractive potential)

- $\exists$ Hawking·Page transition

[cf. Focus Lecture, & AdS/CFT lectures]