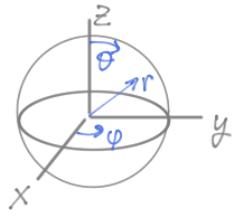


I) Coordinate change

\mathbb{R}^3 metric in Cartesian coords.: $ds^2 = dx^2 + dy^2 + dz^2$



a) coord. change to spherical coords.: (s.t. $x^2 + y^2 + z^2 = r^2$)

$$x = r \sin\theta \cos\varphi \quad \Rightarrow \quad dx = \sin\theta \cos\varphi dr + r \cos\theta \cos\varphi d\theta - r \sin\theta \sin\varphi d\varphi$$

$$y = r \sin\theta \sin\varphi \quad \Rightarrow \quad dy = \sin\theta \sin\varphi dr + r \cos\theta \sin\varphi d\theta + r \sin\theta \cos\varphi d\varphi$$

$$z = r \cos\theta \quad \Rightarrow \quad dz = \cos\theta dr - r \sin\theta d\theta$$

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

b) \rightarrow unit S^2 metric: $ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$

(Note: this can be obtained more directly by setting $r=1$, $dr=0$ above.)

c) \rightarrow generalizing to S^3 : embed into \mathbb{R}^4 :

$$x = \sin\theta \sin\varphi \cos\psi \quad \Rightarrow \quad x^2 + y^2 + z^2 + w^2 = 1 \quad \checkmark$$

$$y = \sin\theta \sin\varphi \sin\psi$$

$$z = \sin\theta \cos\varphi$$

Note: $\theta, \varphi \in [0, \pi]$, $\psi \in [0, 2\pi)$

$$w = \cos\theta$$

$$\therefore ds^2 = dx^2 + dy^2 + dz^2 + dw^2 = d\theta^2 + \sin^2\theta (\sin^2\varphi d\psi^2 + \sin^2\psi d\varphi^2)$$

Here only a single rotational $U(1)$ symmetry is apparent out of the full $SO(4)$.

d) We can do better by pairing rotations in 2-planes of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$:

$$x = \sin\theta \cos\varphi \quad \Rightarrow \quad x^2 + y^2 + z^2 + w^2 = 1 \quad \checkmark$$

$$y = \sin\theta \sin\varphi$$

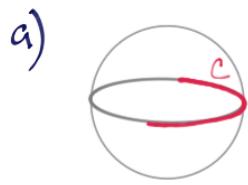
$$z = \cos\theta \cos\psi$$

$$w = \cos\theta \sin\psi$$

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 = \sin^2\theta d\varphi^2 + d\theta^2 + \cos^2\theta d\psi^2$$

\therefore 2 $U(1)$ rot. symmetries

2) Length of curve:



In (x, y, z) coords: $\begin{cases} x(\lambda) = \cos \lambda \\ y(\lambda) = \sin \lambda \\ z(\lambda) = 0 \end{cases}$ w/ $\lambda \in [0, \pi]$

$$\rightarrow \dot{x} = \frac{dx}{d\lambda} = -\sin \lambda$$

$$\rightarrow \dot{y} = \cos \lambda$$

$$\rightarrow \dot{z} = 0$$

$$L(C) = \int \sqrt{|g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu|} d\lambda = \int_0^\pi \sqrt{\dot{x}^2 + \dot{y}^2} d\lambda = \int_0^\pi d\lambda = \pi$$

b) In (r, θ, φ) coords: $r(\lambda) = 1 \rightarrow \dot{r} = 0$
 $\theta(\lambda) = \pi/2 \rightarrow \dot{\theta} = 0$
 $\varphi(\lambda) = \lambda \rightarrow \dot{\varphi} = 1$

$$L(C) = \int \sqrt{\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2)} d\lambda = \int_0^\pi d\lambda = \pi$$

\downarrow \downarrow \downarrow \downarrow
 0 1 0 1

3) Geodesics on S^2 :

a) Not using symmetries: $\ddot{x}^\mu + \Gamma_{\alpha\beta}^\mu \dot{x}^\alpha \dot{x}^\beta = 0$

$$ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2 \rightarrow g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix} \Rightarrow g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sin^2 \theta \end{pmatrix}$$

$$\text{using formula for } \Gamma_{\alpha\beta}^\mu: \quad \Gamma_{\alpha\beta}^\mu = \frac{1}{2} g^{\mu\rho} (\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\alpha\rho} - \partial_\rho g_{\alpha\beta})$$

$$\Gamma_{\theta\theta}^\theta = \frac{1}{2} g^{\theta\theta} (\partial_\theta g_{\theta\theta}) = 0$$

$$\Gamma_{\theta\varphi}^\theta = \frac{1}{2} g^{\theta\theta} (\partial_\theta g_{\varphi\theta} + \partial_\varphi g_{\theta\theta} - \partial_\theta g_{\theta\varphi}) = 0$$

$$\Gamma_{\varphi\varphi}^\theta = \frac{1}{2} g^{\theta\theta} (\partial_\varphi g_{\varphi\theta} \cdot 2 - \partial_\theta g_{\varphi\varphi}) = -\frac{1}{2} \partial_\theta \sin^2 \theta = -\sin \theta \cos \theta$$

$$\Gamma_{\theta\theta}^\varphi = \frac{1}{2} g^{\varphi\varphi} (\partial_\theta g_{\theta\varphi} \cdot 2 - \partial_\varphi g_{\theta\theta}) = 0$$

$$\Gamma_{\theta\varphi}^\varphi = \frac{1}{2} g^{\varphi\varphi} (\partial_\theta g_{\varphi\varphi}) = \frac{1}{2} \frac{1}{\sin^2 \theta} \cdot \partial_\theta \sin^2 \theta = \frac{\cos \theta}{\sin \theta}$$

$$\Gamma_{\varphi\varphi}^\varphi = \frac{1}{2} g^{\varphi\varphi} (\partial_\varphi g_{\varphi\varphi}) = 0$$

OR: Using action shortcut:

$$S = \int (\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2) d\lambda$$

$$\delta\theta: \quad \delta S = \int (2\dot{\theta} \delta\dot{\theta} + 2\sin \theta \cos \theta \dot{\varphi}^2 \delta\theta) d\lambda \stackrel{\text{IP}}{=} -2 \underbrace{\int (\ddot{\theta} - \sin \theta \cos \theta \dot{\varphi}^2) \delta\theta d\lambda}_{=0} = 0 \Rightarrow \Gamma_{\varphi\varphi}^\theta = -\sin \theta \cos \theta$$

$$\delta\varphi: \quad \delta S = \int 2\sin^2 \theta \dot{\varphi} \delta\dot{\varphi} d\lambda \stackrel{\text{IP}}{=} -2 \underbrace{\int \frac{d}{d\lambda} (\sin^2 \theta \dot{\varphi}) \delta\dot{\varphi} d\lambda}_{=0} = \sin^2 \theta \ddot{\varphi} + 2\sin \theta \cos \theta \dot{\theta} \dot{\varphi}$$

$$\Rightarrow \Gamma_{\theta\varphi}^\varphi = \Gamma_{\varphi\theta}^\varphi = \frac{\cos \theta}{\sin \theta}, \quad \text{& all other } \Gamma_{\alpha\beta}^\mu = 0$$

✓

$$\leadsto \text{geodesics given by } \{\theta(\lambda), \varphi(\lambda)\} \text{ satisfying } \begin{cases} \ddot{\theta} - \sin \theta \cos \theta \dot{\varphi}^2 = 0 \\ \ddot{\varphi} + 2 \frac{\cos \theta}{\sin \theta} \dot{\theta} \dot{\varphi} = \frac{d}{d\lambda} (\sin^2 \theta \dot{\varphi}) = 0 \end{cases}$$

e.g. of soln: $\{\theta(\lambda) = \frac{\pi}{2}, \varphi(\lambda) = \lambda\}$, $\{\theta(\lambda) = \lambda, \varphi(\lambda) = \varphi_0\}$ \leadsto great circles on S^2

b) Geodesics on S^2 using constants of motion:

$$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$$

- Geodesic, specified by $\{\theta(\lambda), \varphi(\lambda)\}$, has tangent vector $\xi^\alpha = \dot{\theta} \left(\frac{\partial}{\partial \theta} \right)^\alpha + \dot{\varphi} \left(\frac{\partial}{\partial \varphi} \right)^\alpha$

w/ $\xi^\mu = \left\{ \frac{d}{d\lambda} \theta(\lambda), \frac{d}{d\lambda} \varphi(\lambda) \right\} = \{\dot{\theta}, \dot{\varphi}\}$

- \exists rotational Killing vector $\left(\frac{\partial}{\partial \varphi} \right)^\alpha$, which gives a constant of motion

$$\ell \equiv \xi_\alpha \left(\frac{\partial}{\partial \varphi} \right)^\alpha = \underbrace{\dot{\theta} \left(\frac{\partial}{\partial \theta} \right)_\alpha \left(\frac{\partial}{\partial \varphi} \right)^\alpha}_{g_{\theta\varphi} = 0} + \underbrace{\dot{\varphi} \left(\frac{\partial}{\partial \varphi} \right)_\alpha \left(\frac{\partial}{\partial \varphi} \right)^\alpha}_{g_{\varphi\varphi} = \sin^2\theta} = \sin^2\theta \dot{\varphi}$$

[i.e., even though $\sin^2\theta(\lambda)$ and $\dot{\varphi}(\lambda)$ are individually fns. of λ ,
the combination ℓ is λ -independent \Rightarrow conserved quantity]

- 2nd const. of motion: $\xi_\alpha \xi^\alpha = \dot{\theta}^2 + \sin^2\theta \dot{\varphi}^2 \equiv 1$ for spacelike geod.

$$\Rightarrow 1 = \dot{\theta}^2 + \sin^2\theta \cdot \left(\frac{\ell}{\sin^2\theta} \right)^2 = \dot{\theta}^2 + \frac{\ell^2}{\sin^2\theta}$$

- this gives an ODE for $\theta(\lambda)$: $\frac{d\theta}{d\lambda} = \dot{\theta} = \pm \sqrt{1 - \frac{\ell^2}{\sin^2\theta}}$

\rightarrow Procedure for finding general geod. [characterized by ℓ & parameterized by λ]:

$$\rightarrow \text{integrate: } \lambda(\theta) = \pm \int \frac{d\theta}{\sqrt{1 - \frac{\ell^2}{\sin^2\theta}}}$$

$$\rightarrow \text{invert: } \lambda(\theta) \sim \theta(\lambda)$$

$$\rightarrow \text{integrate } \varphi(\lambda) = \int \dot{\varphi} d\lambda = \int \frac{\ell}{\sin^2\theta(\lambda)} d\lambda$$

} explicitly, we obtain
all great circles of S^2 .

- Alternately, we can find $\varphi(\theta)$ directly by integration:

$$\frac{d\varphi}{d\theta} = \frac{\dot{\varphi}}{\dot{\theta}} = \frac{\pm \ell}{\sin^2\theta \sqrt{1 - \frac{\ell^2}{\sin^2\theta}}} \quad \Rightarrow \quad \varphi(\theta) = \pm \ell \int \frac{d\theta}{\sin^2\theta \sqrt{1 - \frac{\ell^2}{\sin^2\theta}}} = \dots$$

In fact, we can get a closed-form expression: $\varphi(\theta) = \tan^{-1} \frac{\ell \cos\theta}{\sqrt{\sin^2\theta - \ell^2}}$

4) Curvature of S^2

$$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$$

$$\rightarrow \text{Christoffel symbols} \quad \Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\eta} (\partial_\alpha g_{\beta\eta} + \partial_\beta g_{\alpha\eta} - \partial_\eta g_{\alpha\beta})$$

$$\Rightarrow \Gamma^\theta_{\varphi\varphi} = -\sin\theta \cos\theta, \quad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \frac{\cos\theta}{\sin\theta}, \quad \text{all others} = 0$$

a) Riemann tensor $R_{\alpha\beta\gamma}{}^\delta = -\partial_\alpha \Gamma^\delta_{\beta\gamma} + \partial_\beta \Gamma^\delta_{\alpha\gamma} + \Gamma^\mu_{\alpha\gamma} \Gamma^\delta_{\beta\mu} - \Gamma^\mu_{\beta\gamma} \Gamma^\delta_{\alpha\mu}$
 Note summation over μ

Symmetries \Rightarrow non-zero $R_{\alpha\beta\gamma}{}^\delta$ only if $\alpha \neq \beta$ + (for diag. $g_{\mu\nu}$,) $\eta \neq \delta$

so we only need 1 component,

$$R_{\theta\varphi\theta}{}^\varphi = -\partial_\theta \Gamma^\varphi_{\varphi\theta} + \partial_\varphi \Gamma^\varphi_{\theta\theta} + \Gamma^\theta_{\theta\theta} \Gamma^\varphi_{\varphi\theta} + \Gamma^\varphi_{\theta\theta} \Gamma^\varphi_{\varphi\varphi} - \Gamma^\theta_{\varphi\theta} \Gamma^\varphi_{\theta\theta} - \underbrace{\Gamma^\varphi_{\varphi\theta} \Gamma^\varphi_{\theta\varphi}}_{(\frac{\cos\theta}{\sin\theta})^2}$$

$$= 1$$

$$\Rightarrow R_{\varphi\theta\varphi}{}^\theta = g^{\theta\theta} R_{\varphi\theta\varphi\theta} = g^{\theta\theta} R_{\theta\varphi\theta\varphi} = g^{\theta\theta} g_{\varphi\varphi} R_{\theta\varphi\theta}{}^\varphi = g^{\theta\theta} g_{\varphi\varphi} = \sin^2\theta$$

$$+ \text{Simil. } R_{\theta\varphi\varphi}{}^\theta = -R_{\varphi\theta\varphi}{}^\theta = -\sin^2\theta, \quad R_{\varphi\theta\theta}{}^\varphi = -R_{\theta\varphi\theta}{}^\varphi = -1$$

$$+ \text{all other components (e.g. } R_{\theta\theta\theta}{}^\theta \text{)} = 0$$

b) Ricci tensor: $R_{\alpha\beta} = R_{\alpha\gamma\beta}{}^\gamma$

$$R_{\theta\theta} = R_{\theta\theta\theta}{}^\theta + R_{\theta\varphi\theta}{}^\varphi = 1, \quad R_{\theta\varphi} = R_{\varphi\theta} = 0,$$

$$R_{\varphi\varphi} = R_{\varphi\theta\varphi}{}^\theta + R_{\varphi\varphi\varphi}{}^\varphi = \sin^2\theta$$

c) Ricci scalar: $R = R_{\alpha\beta} g^{\alpha\beta} = R_{\theta\theta} \underbrace{g^{\theta\theta}}_1 + R_{\theta\varphi} \underbrace{g^{\theta\varphi}}_0 + R_{\varphi\theta} \underbrace{g^{\varphi\theta}}_1 + R_{\varphi\varphi} \underbrace{g^{\varphi\varphi}}_{\sin^2\theta} \frac{1}{\sin^2\theta}$

$$R = 2$$

Hence the sphere has constant positive curvature.