

1) Geods in Schwarzschild :

a) Let's find geods in $ds^2 = -f(r) dt^2 + h(r) dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$:
 ↳ we want to solve for $(t(r), r(r), \theta(r), \varphi(r))$

tangent vector in coord. frame $\vec{p}^a = \dot{t} \left(\frac{\partial}{\partial t} \right)^a + \dot{r} \left(\frac{\partial}{\partial r} \right)^a + \dot{\theta} \left(\frac{\partial}{\partial \theta} \right)^a + \dot{\phi} \left(\frac{\partial}{\partial \phi} \right)^a$

spherical symmetry \Rightarrow wlog set $\Theta(\lambda) = \frac{\pi}{2}$, $\dot{\Theta} = 0 \rightsquigarrow$ equatorial orbits

$$\stackrel{*}{\equiv} \frac{d}{d\lambda} \quad \text{w/ } \lambda = \text{affine param.} \Rightarrow p_a p^a \stackrel{*}{=} \chi = \begin{cases} 1 & \text{for spacelike geodesics} \\ 0 & \text{for null} \\ -1 & \text{for timelike} \end{cases}$$

const. of motion

Since $\left(\frac{\partial}{\partial t}\right)^a + \left(\frac{\partial}{\partial \psi}\right)^a$ are Killing vectors

$$\text{since } p^a \nabla_a x = p^a \nabla_a (p_b p^b) = 2 p_b \underbrace{p^a \nabla_a p^b}_0 = 0$$

$\Rightarrow \exists$ conserved quantities (constants of motion).

$$\text{Here (for \((*)\)), } E = -\dot{t} \underbrace{\left(\frac{\partial}{\partial t} \right)_{\alpha} \left(\frac{\partial}{\partial t} \right)^{\alpha}}_{g_{tt}} + 0 = \dot{t} f(r) , \quad L = r^2 \underbrace{\sin^2 \theta}_{1} \dot{\phi} = r^2 \dot{\phi}$$

$$\text{So } \kappa = p_a p^a = -f(r) \dot{t}^2 + h(r) \dot{r}^2 + r^2 \dot{\varphi}^2 = -\frac{E^2}{g(r)} + h(r) \dot{r}^2 + \frac{L^2}{r^2} \rightarrow \text{"radial eqn"}$$

This has the form of particle in 1-d effective potential: $\frac{1}{2} \dot{r}^2 + V_{\text{eff}}(r) = 0$
 K.E. P.E.

$$V_{\text{eff}}(r) = \frac{1}{2\hbar^2} \left(-\kappa - \frac{E^2}{f(r)} + \frac{L^2}{r^2} \right)$$

(wlog, total energy = 0
for convenience)

Now that we have the general form, it's easy to specialize to Schwarzschild:

$$f(r) = \frac{1}{h(r)} = 1 - \frac{2M}{r} , \quad V_{\text{eff}}(r) = - \frac{(E^2 + \chi)}{2} + \chi \frac{M}{r} + \frac{L^2}{2r^2} - \frac{L^2 M}{r^3}$$

\therefore To find explicit expression for geodesics in Schw, we just need to integrate:

$$\text{Given } M, L, E, \kappa, \quad V_{\text{eff}}(r) = -\frac{(E^2 + \kappa)}{2} + \kappa \frac{M}{r} + \frac{L^2}{2r^2} - \frac{L^2 M}{r^3}$$

$$\begin{aligned}\dot{r} &= \pm \sqrt{-2V_{\text{eff}}(r)} = \frac{dr}{d\lambda} \quad \rightarrow \quad \lambda = c \pm \int \frac{dr}{\sqrt{-2V_{\text{eff}}(r)}} \quad \text{+ invert to get } r(\lambda) \\ \dot{t} &= E/s(r) = E / \left(1 - \frac{2M}{r(\lambda)}\right) \quad \rightarrow \quad t(\lambda) = c + \int \frac{E d\lambda}{1 - \frac{2M}{r(\lambda)}} \\ \dot{\phi} &= L/r^2 \quad \rightarrow \quad \phi(\lambda) = C + \int \frac{L}{r(\lambda)^2} d\lambda\end{aligned}$$

We can also get the ST trajectories directly ($\&$ find a.p. λ later if we need to)

$$\text{eg. } \frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = \frac{\pm E}{\hbar r \sqrt{-2V_{\text{eff}}(r)}} \quad \Rightarrow \quad t(r) = \int \dots dr \quad \text{etc.}$$

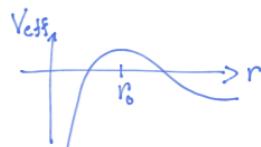
To do this in full generality is however messy $\&$ unilluminating... — but usually we don't need to.

We can read off most of what we need directly from V_{eff} ...

b) Null geodesics:

$$V_{\text{eff}}(r) = -\frac{E^2}{2} + \frac{L^2}{2r^2} - \frac{L^2 M}{r^3}$$

$$V'_{\text{eff}}(r) = -\frac{L^2}{r^4} (r - 3M) = 0$$



$$\Rightarrow r_0 = 3M \rightarrow \text{unstable circular orbit!}$$

$$V_{\text{eff}}(r_0) = -\frac{E^2}{2} + \frac{L^2}{2 \cdot 9M^2} - \frac{L^2 M}{27M^3} = -\frac{E^2}{2} + \frac{L^2}{2 \cdot 27M^2} = 0 \quad \Rightarrow \quad \frac{L}{E} = 3\sqrt{3}M$$

c) Radial null geods:

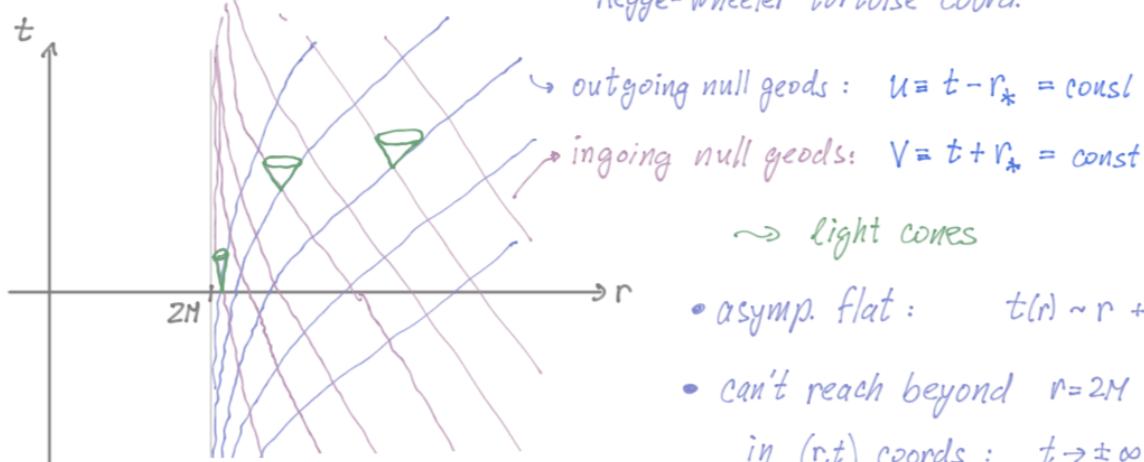
- Although Schw. is 4-d, due to spher. sym. we can focus on the (r,t) plane \rightarrow effectively 2-d.

$$\therefore \text{Consider } ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}}$$

+ adapt coords to radial null geods $\hookrightarrow ds^2=0 \Rightarrow \frac{dt}{dr} = \pm \frac{1}{1-\frac{2M}{r}}$

$$\Rightarrow t = \pm r_* + \text{const.}, \text{ w/ } r_* \equiv r + 2M \ln\left(\frac{r}{2M}-1\right) \Rightarrow dr_* = \frac{dr}{1-\frac{2M}{r}}$$

\hookrightarrow "Regge-Wheeler tortoise coord."



- asymp. flat: $t(r) \sim r + \text{const}$ as $r \rightarrow \infty$
- can't reach beyond $r=2M$
in (r,t) coords: $t \rightarrow \pm\infty$ as $r \rightarrow 2M$

d)

geods reach $r=2M$ @ finite a.p.:

$$\text{q. null geods: } 0 = p_a p^a = -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1-\frac{2M}{r}} = \frac{-E^2 + \dot{r}^2}{1-\frac{2M}{r}} \Rightarrow r(\lambda) = \pm \lambda E + \text{const.}$$

λ is finite for $\swarrow r \rightarrow 2M$.

2) AdS:

a) Embedding for global coords:

We want a surface $-X_{-1}^2 - X_0^2 + X_1^2 + \dots + X_3^2 = -\ell^2$
 in flat $(2, d-1)$ ST: $ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + \dots + dX_3^2$
 cluster \downarrow to get static + spher. sym. ansatz:

$$\begin{aligned} X_{-1} &= \sqrt{r^2 + \ell^2} \sin \frac{t}{\ell} \\ X_0 &= \sqrt{r^2 + \ell^2} \cos \frac{t}{\ell} \end{aligned} \quad \left. \begin{aligned} -X_{-1}^2 - X_0^2 &= -r^2 - \ell^2 \end{aligned} \right\}$$

$$\begin{aligned} X_1 &= r \sin \theta \cos \varphi \\ X_2 &= r \sin \theta \sin \varphi \\ X_3 &= r \cos \theta \end{aligned} \quad \left. \begin{aligned} X_1^2 + X_2^2 + X_3^2 &= r^2 \end{aligned} \right\}$$

$\rightarrow dX_3 = \cos \theta dr - r \sin \theta d\theta$
 etc.

$$ds^2 = -\left(\frac{r^2}{\ell^2} + 1\right) dt^2 + \frac{dr^2}{\left(\frac{r^2}{\ell^2} + 1\right)} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$

b) Radial timelike geods.

It's easier to change coords slightly: $r = \tan \varphi$

+ WLOG set AdS length scale $\ell \equiv 1$ (we can reinstate ℓ later using dimensional analysis.)

Then $ds^2 = \frac{1}{\cos^2 \varphi} (-dt^2 + d\varphi^2 + \sin^2 \varphi d\Omega^2)$
 ↓ ignore:

CoMs: energy $E = \frac{\dot{t}}{\cos^2 \varphi}$, ang. mom. $L = 0$ since radial

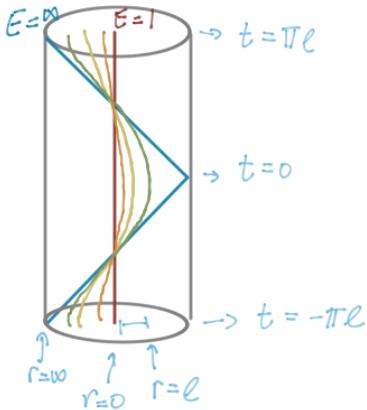
$$K = -1 = -\frac{\cos^4 \varphi E^2 + \dot{\varphi}^2}{\cos^2 \varphi} \Rightarrow V_{\text{eff}}(\varphi) = \cos^2 \varphi (1 - \cos^2 \varphi E^2)$$

\Rightarrow geods are bound in φ : $\varphi_{\max} = \cos^{-1} \frac{1}{E}$

$$t(\beta) = \int \frac{dt}{ds} d\beta = \int \frac{E \cos^2 \beta \frac{d\beta}{ds}}{\cos \beta \sqrt{\cos^2 \beta E^2 - 1}} = \dots = \sin^{-1} \left(\frac{E}{\sqrt{E^2 - 1}} \sin \beta \right)$$

$$\Rightarrow g(t) = \sin^{-1} \left[\frac{\sqrt{E^2 - 1}}{E} \sin t \right]$$

reinstating AdS scale
↓
 $\hookrightarrow \therefore$ periodicity $2\pi l \quad \forall E:$



c) Circular orbits:

Recall (1a): for $ds^2 = -f(r) dt^2 + h(r) dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$

the effective potential is $V_{\text{eff}}(r) = \frac{1}{2h(r)} \left(-\kappa - \frac{E^2}{f(r)} + \frac{L^2}{r^2} \right)$

$$\begin{aligned} \text{So for } f(r) = \frac{1}{h(r)} = r^2 + 1 \quad 2V_{\text{eff}}(r) &= (r^2 + 1) \left(-\kappa - \frac{E^2}{r^2 + 1} + \frac{L^2}{r^2} \right) = \\ &= -\kappa r^2 + (L^2 - E^2 - \kappa) + \frac{L^2}{r^2} \end{aligned}$$

$$\therefore 2V'_{\text{eff}}(r) = -2\kappa r - \frac{2L^2}{r^3}$$

this can = 0 only if $\kappa = -1$, i.e. only timelike geods. can have circular orbits.

Since $V''_{\text{eff}}(r) = -\kappa + \frac{6L^2}{r^4} > 0$ if $\kappa = -1$, [↑] such circular orbits are stable.

d) AdS boundary metric:

$$ds^2 = -(r^2 + l^2) dt^2 + \frac{dr^2}{r^2 + l^2} + r^2 d\Omega^2$$

@ const. $r = r_\infty$, $\left. ds^2 \right|_{r_\infty} = -(r_\infty^2 + l^2) dt^2 + r_\infty^2 d\Omega^2$

$$\therefore \lim_{r_\infty \rightarrow \infty} \left. \frac{1}{r_\infty^2} ds^2 \right|_{r_\infty} = -dt^2 + d\Omega^2$$

This is called Einstein static universe (ESU)

\therefore Global AdS boundary metric is ESU.

(N.B. If we kept l , we could get rid of it by rescaling t).