SUSY AND TWIST: EXERCISE

1. Exercise: day 1

(1) Consider the unitary representation of $d = 2, N = (1,1)$ SUSY algebra

$$[Q_+, Q_+] = -\sqrt{-1}(H + P), \quad [Q_-, Q_-] = -\sqrt{-1}(H - P), \quad [Q_+, Q_-] = \sqrt{-1}Z$$

with the unitarity condition

$$Q_+^\dagger = Q_+, \quad Q_-^\dagger = Q_-.$$

Here Z is a constant (central charge). H is the Hamiltonian operator, and P is the momentum operator. Eigenvalues of H are the energies. Assume E is an eigenvalue of H. Show that unitarity implies the following BPS bound

$$E \geq |Z|.$$

(2) In this exercise we consider spin group in the Euclidean space $\mathbb{R}^3$. Let $\mathbb{H}$ be the quaternions

$$\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$$

with the quaternion relations

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad ki = -ik = j, \quad jk = -kj = i.$$

Recall the conjugate and norm for $q = x_0 + x_1 i + x_2 j + x_3 k$

$$\bar{q} = x_0 - x_1 i - x_2 j - x_3 k, \quad |q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

$S = \{q \in \mathbb{H} | |q| = 1\}$ forms a group. The imagery part of $\mathbb{H}$ is defined by

$$\mathrm{im}(\mathbb{H}) = \{q \in \mathbb{H} | \bar{q} = -q\} = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k.$$

a) Show that there is a group isomorphism $SU(2) \simeq S$ under the identification

$$i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}, \quad k = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$$

b) $\mathrm{im}(\mathbb{H})$ defines a natural representation of S by

$$q : r \to qr\bar{q}, \quad q \in S, r \in \mathrm{im}(\mathbb{H}).$$

Show that under the identification of $\mathbb{R}^3$ with $\mathrm{im}(\mathbb{H})$ and $SU(2)$ with S, the above representation defines a group homomorphism $SU(2) \to SO(3)$. This homomorphism is a double cover (and universal cover). This shows that $Spin(3) \simeq SU(2)$.

(3) Recall the definition of integration for fermionic (odd) variables. Given an odd variable θ, with $\theta^2 = 0$, we define

$$\int d\theta(a + \theta b) = b$$

where a, b don't depend on θ. For n odd variables, $\int d\theta^1 d\theta^2 \cdots d\theta^n$ can be defined by $\int d\theta^1 \cdots \int d\theta^n$.

(a) Show that: $\int d\theta(\partial_\theta f) = 0$ and $\int d\theta^1 d\theta^2(-) = -\int d\theta^2 d\theta^1(-)$.

(b) Let $A_{ij} = -A_{ji}$ be a skew-symmetric matrix. Evaluate the following integral

$$\int d\theta^1 \cdots d\theta^n e^{\sum_{ij} A_{ij}\theta^i\theta^j}.$$