1 Lecture 1

1. Consider the operator
$$h = \int_0^\infty d\tau\, Q^\dagger e^{-\tau H}\,.$$
Assuming it exists, show that
$$\{Q, h\} = 1 - P\,,$$
where $P : \Omega \to \mathcal{H}$ is the projection onto supersymmetric ground states.

2. Suppose there are $\mathcal{N}$ self-adjoint supercharges Q_j obeying
$$\{\widetilde{Q}_i, \widetilde{Q}_j\} = \delta_{ij} H \qquad j = 1, \ldots, \mathcal{N}\,.$$
   (a) What is the group of outer automorphisms transforming the $\widetilde{Q}_j$'s?
   (b) Show that $\mathcal{N} = 2$ is equivalent to the superalgebra considered in the lectures where
$$Q = \widetilde{Q}_1 + i\widetilde{Q}_2\,.$$

3. A 3d $\mathcal{N} = 2$ theory in euclidean $\mathbb{R}^3$ has supersymmetry algebra
$$\{Q_\alpha, \widetilde{Q}_\beta\} = (\sigma^i)_{\alpha\beta} P_i \tag{1.1}$$
where Q_α, $\widetilde{Q}_\alpha$ are euclidean $SO(3)$ spinors and $(\sigma^i)_\alpha{}^\beta$ are the Pauli matrices. There is an $SO(2)_R$ R-symmetry transforming Q_α, $\widetilde{Q}_\alpha$ with weight $+1$, -1.
   (a) Identify the supercharges commuting with the diagonal combination of $SO(2)_R$ and the subgroup $SO(2)_{12} \subset SO(3)$ of rotations in the $x^{1,2}$-plane.
   (b) Show that they generate a SQM in the x^3-direction.
   (c) Why is this relevant for performing a topological twist on $\mathbb{R} \times \Sigma$ where Σ is a closed orientable Riemann surface?

The following is an extended questions whose solution is mostly contained in section 10.4.3 of the 'Mirror Symmetry' book.

4. Show that the $\mathcal{N} = 4$ superalgebra is equivalent to

$$\{Q_\alpha, Q_\beta\} = 0 \qquad \{Q_\alpha, Q_\beta^\dagger\} = \delta_{\alpha\beta} H \qquad \{Q_\alpha^\dagger, Q_\beta^\dagger\} = 0$$

for some linear combinations Q_+, Q_-.

(a) Construct a representation of the $\mathcal{N} = 4$ superalgebra from the data of a compact smooth Kähler manifold X.

(b) What subgroup of the outer automorphism group $O(4)$ is a symmetry of such a supersymmetric quantum mechanics? Don't forget the Lefschetz action!

(c) Identify the $\mathcal{N} = 2$ subalgebra corresponding to the Riemannian sigma model considered in the lectures.