4. Show that the $\mathcal{N} = 4$ superalgebra is equivalent to

$$\{Q_\alpha, Q_\beta\} = 0 \qquad \{Q_\alpha, Q^\dagger_\beta\} = \delta_{\alpha\beta} H \qquad \{Q^\dagger_\alpha, Q^\dagger_\beta\} = 0$$

for some linear combinations Q_+, Q_-.

(a) Construct a representation of the $\mathcal{N} = 4$ superalgebra from the data of a compact smooth Kähler manifold X.

(b) What subgroup of the outer automorphism group $O(4)$ is a symmetry of such a supersymmetric quantum mechanics? Don't forget the Lefschetz action!

(c) Identify the $\mathcal{N} = 2$ subalgebra corresponding to the Riemannian sigma model considered in the lectures.

2 Lecture 2

1. Consider a supersymmetric quantum mechanics with $X = S^3$ with the round metric and flavour symmetry $G = U(1)$ acting by freely rotating the fibers of the Hopf fibration. Introducing polar coordinates (θ, ζ, ϕ) with $0 \leq \theta \leq \frac{\pi}{2}$ and $\zeta, \phi \in \mathbb{R}/2\pi\mathbb{Z}$, the volume form is

$$\omega_3 = \sin\theta \cos\theta \, d\theta \wedge d\zeta \wedge d\phi$$

and the vector field generating the $U(1)$ action is $V = \partial_\zeta + \partial_\phi$.

(a) Show that $\omega_2 := \gamma \cdot \omega_3 = 2\pi \sin\theta \cos\theta \, d\theta \wedge (d\zeta - d\phi)$.

(b) Show that $\omega_2 = d\omega_1$ where $\omega_1 = \pi(\sin^2\theta \, d\zeta + \cos^2\theta \, d\phi)$.

(c) Show that $\omega_0 := \gamma \cdot \omega_1 = 2\pi^2$.

(d) Explain why there is a nontrivial higher operation $H_1(U(1))^2 : H^3(S^3) \to H^0(S^3)$.