## 1. Correlation functions from BV formalism

Consider a massive free scalar field theory on some compact Riemannian manifold $M$. The field is $\phi$, the anti-field is $\psi$, and the Lagrangian is

$$\int \phi(\mathrm{D} + m^2)\phi \tag{1}$$

where D is the Laplacian with non-negative spectrum.

For a smooth function $f \in C_c^\infty(U)$ with compact support, on some open $U \subset M$, we have an observable $\mathcal{O}_f$ defined by

$$\mathcal{O}_f(\phi) = \int_U f\phi \mathrm{d}Vol \tag{2}$$

(we use the Riemannian volume form). Similarly, we have an observable $\mathcal{O}_f^*$ of cohomological degree $-1$ defined by

$$\mathcal{O}_f^*(\psi) = \int_u f\psi \mathrm{d}Vol. \tag{3}$$

In class, we define the complex $\mathrm{Obs}^q(U)$ of quantum observables on $U$ to be the algebra of polynomials in $\mathcal{O}_f$, $\mathcal{O}_g^*$ and a variable $\hbar$, as $f, g$ range over $C_c^\infty(U)$. The differential is

(4)

$$Q\left(\mathcal{O}_{f_1}\dots\mathcal{O}_{f_m}\mathcal{O}_{g_1}^*\dots\mathcal{O}_{g_n}^*\right) = \sum_{i=1}^{n}(-1)^{i-1}\mathcal{O}_{f_1}\dots\mathcal{O}_{f_m}\mathcal{O}_{g_1}^*\dots\mathcal{O}(\mathrm{D}+m^2)g_i\dots\mathcal{O}_{g_n}^*$$
$$+\hbar\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{j-1}\hbar\left(\int f_i g_j \mathrm{d}Vol\right)\mathcal{O}_{f_1}\dots\widehat{\mathcal{O}_{f_i}}\mathcal{O}_{f_m}\mathcal{O}_{g_1}^*\dots\widehat{\mathcal{O}_{g_j}^*}\mathcal{O}_{g_n}^*$$

The first term comes from the classical Lagrangian, the second from the BV Laplacian.

**Question 1.** *By using a spectral sequence associated to the increasing filtration on the complex of observables by polynomial degree in the generators $\mathcal{O}$, $\mathcal{O}^*$, show that*

$$H^*(\mathrm{Obs}^q(M)) = \mathbb{C}[\hbar] \tag{5}$$

*as $\mathbb{C}[\hbar]$ modules.*

Normalize this isomorphism so that the observable $1 \in \mathrm{Obs}^q(M)$ is sent to $1 \in \mathbb{C}[\hbar]$.

If $U_1, \ldots, U_n \subset M$ are disjoint opens, define the correlation functions to be the map

$$\langle - \rangle : H^*(\mathrm{Obs}^q(U_1)) \otimes \cdots \otimes H^*(\mathrm{Obs}^q(U_n)) \to H^* \mathrm{Obs}^q(M)) \cong \mathbb{C}[\hbar] \tag{6}$$

where the first map is the factorization product.

**Question 2.** *Let $G(x,y)$ be the distribution on $M^2$ characterized by the fact that*

$$(\mathrm{D}_x + m^2) G(x,y) = \delta_{x=y} \tag{7}$$

*(of course, $G(x,y)$ is the Green's function).*

*If $U, V \subset M$ are disjoint opens, and $f \in C_c^\infty(U)$, $g \in C_c^\infty(V)$, show that*

$$\langle \mathcal{O}_f, \mathcal{O}_g \rangle = \hbar \int_{(x,y) \in M \times M} f(x) g(y) G(x,y) \mathrm{d}Vol_x \mathrm{d}Vol_y. \tag{8}$$

This is the usual expression in physics for correlation functions. You should prove this by using the Green's function $G(x,y)$ to make the product $\mathcal{O}_f \mathcal{O}_g \in \mathrm{Obs}^q(M)$ cohomologous to a multiple of $\hbar \in \mathrm{Obs}^q(M)$.

**Question 3.** *Derive Wick's lemma, which is an expression for the expectation value of the product $\mathcal{O}_{f_1} \ldots \mathcal{O}_{f_n}$ for $f_i \in C_c^\infty(U_i)$, $U_i$ disjoint opens (disjointness is not really necessary here).*