Noether's Theorem states that every global symmetry of an action is associated to a conserved current J. The current is local operator (supported in the formal neighborhood of a point p) valued in the cotangent space to that point

$$J \in \mathrm{Ops}(p) \otimes T_p^*(\mathbb{R}^d) \quad \text{(for spacetime } \mathbb{R}^d\text{)}\,. \tag{1}$$

One usually lets p vary, and thinks of J as a 1-form on spacetime. Being conserved means that

$$\langle \cdots (d * J) \cdots \rangle = 0 \tag{2}$$

in every correlation function, where '$\cdots$' denote other operators whose support is disjoint from that of J. In Kevin's formalism, one would say that $d * J$ is cohomologous to zero as an element of the factorization algebra.

In quantum mechanics, spacetime is $\mathbb{R}$, and $\mathcal{Q} = *J \in \mathrm{Ops}$ is an ordinary local operator. It is called the conserved charge associated to a global symmetry. In any correlation function, we have

$$\langle \cdots \partial_t \mathcal{Q}(t) \cdots \rangle = 0\,. \tag{3}$$

In other words, correlation functions involving $\mathcal{Q}(t)$ are locally independent of the insertion point t. Conserved charges are *topological* local operators.

Noether's theorem in quantum mechanics works like this. (The argument generalizes in a completely standard way to get J higher dimensions.)

Let v be a vector field on the space of fields $\mathcal{F}$ that generates a global symmetry. Being a global symmetry means that the action is invariant

$$\mathcal{L}_v S = 0\,.$$

Now scale v by a constant α. Then $\mathcal{L}_{(\alpha v)} S = \alpha \mathcal{L}_v S = 0$ still.

But what if we allow $\alpha = \alpha(t)$ to depend on time? Then $\mathcal{L}_{\alpha v} S$ need no longer be zero, but may instead have a linear dependence on $\dot\alpha$. Perhaps after an integration by parts, we should find that $\mathcal{L}_{\alpha v} S$ is a contraction of $\dot\alpha$ and some local operator $\mathcal{Q}$:

$$\mathcal{L}_{\alpha v} S = -\int dt\, \dot\alpha\, \mathcal{Q} = \int dt\, \alpha\, \partial_t \mathcal{Q}\,. \tag{4}$$

On the other hand, if we impose the solutions of the equations of motion, any variation of the action (including this one) must vanish. Since $\alpha(t)$ can be chosen arbitrarily, it follows that

$$\partial_t \mathcal{Q}\big|_{EOM} = 0\,. \tag{5}$$

In the quantum theory, this becomes an equation satisfied inside correlation functions (3).

In the Hilbert space/operator formalism, conserved charges represent the original global symmetry on the Hilbert space $\mathcal{H}$. For example, if v generates the action of a group G, then v is really valued in $\mathfrak{g}^*$, and so is the conserved charge $\mathcal{Q}$. Since $\mathcal{Q}$ is time-independent (meaning it commutes with the Hamiltonian...) it acts the same way on $\mathcal{H}$ no matter where it is inserted. Given two elements $a, b \in \mathfrak{g}$, we expect the action of $\mathcal{Q}$ on the Hilbert space to satisfy

$$[\langle a, \mathcal{Q}\rangle, \langle b, \mathcal{Q}\rangle] = \mathcal{Q}_{[a,b]}\,. \tag{6}$$

1. Consider the action for a free complex fermion, with a mass term

$$S = \int dt \left[\overline{\psi} \partial_t \psi + m \overline{\psi} \psi \right] . \tag{7}$$

a) What are the equations of motion for ψ and $\overline{\psi}$? What are the local operators? What is the Hilbert space, and how do $\overline{\psi}, \psi$ act on it?

b) Consider the $U(1)$ global symmetry that rotates $\psi, \overline{\psi}$ with opposite phases (with weights ± 1). Note that it preserves the action. Write down a vector field v on the space of fields that generates this symmetry, and use Noether's theorem to find the corresponding conserved charge $\mathcal{Q}$. How does the local operator $\mathcal{Q}$ act on the Hilbert space?

c) Consider the $\mathbb{R}$ symmetry that shifts the time coordinate $t \mapsto t + a$. What is its conserved charge H? Here and in any quantum mechanics, the conserved charge associated to time translations is the Hamiltonian. (This is a good way to define and compute the Hamiltonian.)

How does H act on the Hilbert space? Check that $[H, \mathcal{Q}] = 0$.

What happens in the limit $m \to 0$? You should find that the entire theory becomes topological, meaning that not just $\mathcal{Q}$ but *every* local operator commutes with H in this limit.