1. 3rd Day

Question 1. *Consider an n-dimensional vector space V with Lebesgue measure $\mu_{Leb} = dx^1 \wedge \cdots \wedge dx^n$. For $f \in \mathcal{O}(V)$, show*

$$\Delta_{Leb}(f\partial_1 \wedge \cdots \wedge \partial_n) = \sum_{i=1}^{n} (-1)^{i-1} (\partial_i f)\partial_1 \wedge \cdots \wedge \widehat{\partial_i} \wedge \cdots \wedge \partial_n$$

and hence $\Delta_{Leb} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x^i \partial \xi_i}$ is a second-order differential operator.

Question 2. *Consider a (-1)-shifted vector space (W, ω) with $\omega \in (\wedge^2 W^*)[-1]$. Identify the corresponding Poisson bivector K in as an element. of $(\mathrm{Sym}^2 W)[1]$. Show that a BV differential Δ on $\mathcal{O}(W)$ can be defined as the second-order operator given by contracting with K. This defines the Poisson bracket $\{-, -\}$ as well from the BD algebra axiom*

$$d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot db + \hbar\{a, b\}.$$

Question 3. *Consider a free scalar field theory, where M is a compact Riemannian manifold, $\phi \in C^\infty(M)$, $\psi \in \Omega^d(M)$, and $S(\phi) = \int_M \phi D\phi$. Consider*

(a) *For $\alpha \in \Omega^d(M)$, we define a linear functional $O_\alpha \colon \mathcal{E} \to \mathbb{R}$ by $O_\alpha(\phi, \psi) = \int_M \phi\alpha$.*
(b) *For $f \in C^\infty(M)$, we define a linear functional $O_f \colon \mathcal{E} \to \mathbb{R}$ by $O_f(\phi, \psi) = \int_M f\psi$. This is a functional of cohomological degree -1.*
(c) *For $T \in \Omega^d(M \times M)$, we define a quadratic functional $O_T \colon \mathcal{E} \to \mathbb{R}$ by $O_T(\phi, \psi) = \int_{M\times M} \phi(x_1)\psi(x_2)T(x_1, x_2)$. This is also of degree -1.*

1) *What are local functionals? What if α, f, T are distributional?*
2) *Compute $K_0 \in \mathrm{Sym}^2(\overline{\mathcal{E}}\widehat{\otimes}\overline{\mathcal{E}})[1]$ and prove $\{O_\alpha, O_f\} = \int_M f\alpha \in \mathbb{R}$. Note that this would be ill-defined if both α, f are distributional.*
3) *Find an example of T where the associated functional O_T is local but ΔO_T is ill-defined.*

Question 4. *We defined $W(P, I) = \hbar \log(\exp(\hbar\partial_P) \exp(I/\hbar))$. It is known that one can write $W(P, I) = \sum_\gamma \frac{\hbar^{g(\gamma)}}{\mathrm{Aut}\,\gamma} w_\gamma(P, I)$, where the sum is over all connected graphs with propagator P on edges and interaction I on vertices and $g(\gamma)$ is the genus of the graph γ. Can you prove it?*

Question 5. *Show that a functional $I \in \mathcal{O}^+(\mathcal{E})[\![\hbar]\!]$ satisfies the scale ϵ QME if and only if $W(P(\epsilon, L), I)$ satisfies the scale L QME.*

1.1. Idea of Effective Field Theory.

The idea is that the story we are telling is happening at the infinite energy scale. Even physically, no experiment happens at the infinite energy scale. In other words, anything physical can be observed at a finite energy scale, say Λ.

The idea of effective field theory follows from an extended version of the action principle, namely, there should exist an *effective action* $S[\Lambda] = S_{\text{free}} + I[\Lambda]$ for an energy scale Λ such that it captures all the physics happening in the energy scale $< \Lambda$. For concreteness, for scalar field theory, we let $C^\infty(M)_{\leq\Lambda}$ be the space of functions that is generated by eigenfunctions with eigenvalue $\leq \Lambda$ with respect to the Laplacian operator D.

Let us apply this idea to our favorite heuristic formula

$$\langle O_1, \cdots, O_n \rangle = \langle O_1, \cdots, O_n \rangle_\infty = \int_{\phi \in C^\infty(M)} e^{S(\phi)/\hbar} O_1(\phi) \cdots O_n(\phi) D\phi$$

and replace $C^\infty(M)$ by $C^\infty(M)_{\leq \Lambda}$ and think of the space of observables $\mathrm{Obs}_{\leq \Lambda}$ consisting of functionals $C^\infty(M)_{\leq \Lambda} \to \mathbb{R}$, which are extended to $C^\infty(M)$ by the projection $C^\infty(M) \to C^\infty(M)_{\leq \Lambda}$. The upshot is that if $O_i \in \mathrm{Obs}_{\leq \Lambda} = \mathcal{O}(C^\infty(M)_{\leq \Lambda}) \subset \mathrm{Obs} = \mathcal{O}(C^\infty(M))$, then one should have

$$\langle O_1, \cdots, O_n \rangle_\infty = \langle O_1, \cdots, O_n \rangle_\Lambda := \int_{\phi \in C^\infty(M)_{\leq \Lambda}} e^{S[\Lambda](\phi)/\hbar} O_1(\phi) \cdots O_n(\phi) D\phi.$$

In particular, if $\Lambda_L \leq \Lambda_H$, then one has $\mathrm{Obs}_{\leq \Lambda_L} \hookrightarrow \mathrm{Obs}_{\leq \Lambda_H}$. In this case, if $O_i \in \mathrm{Obs}_{\leq \Lambda_L}$, one should have an equality

$$\begin{aligned} & \int_{\phi \in C^\infty(M)_{\leq \Lambda_L}} e^{S[\Lambda_L](\phi)/\hbar} O_1(\phi) \cdots O_n(\phi) D\phi \\ = & \int_{\phi \in C^\infty(M)_{\leq \Lambda_H}} e^{S[\Lambda_H](\phi)/\hbar} O_1(\phi) \cdots O_n(\phi) D\phi \\ = & \int_{\phi_L \in C^\infty(M)_{\leq \Lambda_L}} \int_{\phi_H \in C^\infty(M)_{(\Lambda_L, \Lambda_H]}} e^{S[\Lambda_H](\phi_L + \phi_H)/\hbar} O_1(\phi) \cdots O_n(\phi) D\phi. \end{aligned}$$

This leads to the equality

$$S[\Lambda_L](\phi_L) = \hbar \log \left(\int_{\phi_H \in C^\infty(M)_{(\Lambda_L, \Lambda_H)}} e^{S[\Lambda_H](\phi_L + \phi_H)/\hbar} D\phi \right),$$

called *renormalization group equation* or *RG equation* for short. One can read the RG equation as saying that for $\Lambda_L \leq \Lambda_H$, the effective action $S[\Lambda_L]$ at a lower energy scale Λ_L can be obtained from the effective action $S[\Lambda_H]$ at a higher energy scale Λ_H by integrating out fields with energy between Λ_L and Λ_H. By construction, renormalization group equation would ensure that correlation functions are well-defined without depending on the arbitrary choice of the energy scale Λ the observables are considered to be measured at.

Now we recall another fundamental principle of quantum field theory which is locality saying that interactions happen at points. However, our energy scale description of RG equation is not adequate to capture locality faithfully; after all, the eigenvalues of a Laplacian are not local on spacetime. To remedy this defect, we instead work with a length scale $L > 0$, rather than $\Lambda < \infty$, thinking of L as $\frac{1}{\Lambda}$. Then $I[L]$ is supposed to be understood as a "scale L effective interaction" that encodes everything happening at length scale greater than L.

The renormalization group equation tells us that for $L_2 > L_1$, the effective interaction $I[L_2]$ is obtained from $I[L_1]$ by allowing particles to evolve with proper time between L_1 and L_2 and to interact by $I[L_1]$. In particular, a functional $I[\infty]$ should encode correlation functions $\langle \phi(x_1), \cdots, \phi(x_n) \rangle$ as the sum of graphs which have external vertices with labeling $x_1, \cdots, x_n$ and internal vertices labeled by $I[\infty]$. Note that the relevant diagram for $I[\infty]$ would be much simpler just because anything happening in length scale $< \infty$ would be already encoded in $I[\infty]$, as opposed to being present in the diagram. This is also compatible with the interpretation that anything would look simple through an infinitely zoomed-out lens.