3. EXERCISE: DAY 3

We consider the Mathai-Quillen of SUSY quantum mechanics. The space of fields is the loop space

$$LX = \{\phi : S^1 \to X\}.$$

The bundle E on LX we are interested in is the tangent bundle of LX

$$E|_\phi = \Gamma(S^1, \phi^* TX) = T_\phi LX.$$

Let t denote the periodic coordinate on S^1. There is a canonical section s of E by

$$s(\phi) = \frac{d\phi}{dt}$$

i.e. the tangent map. The moduli space M of the zero locus of s consists of constant maps, i.e.

$$M \cong X.$$

The Riemannian metric g on X gives a natural metric on LX by

$$\langle V, W\rangle = \int_{S^1} dt g(V, W) = \int dt g_{ij}(\phi) V^i W^j, \quad V = V^i \partial_i, W = W^i \partial_i \in \Gamma(S^1, \phi^* TX).$$

(1) Apply the Mathai-Quillen formalism to find the following expression of localized Euler class of E

$$e_s(E) = \int_{LX} d\phi d\psi d\bar{\psi} dB e^{-\frac{1}{\hbar}\delta\Psi}$$

Here $\psi \in \Gamma(S^1, \phi^* TX)$, $\bar{\psi} \in \Gamma(S^1, \phi^*(TX)^\vee)$ are fermions, $B \in \Gamma(S^1, \phi^*(TX)^\vee)$ is a boson. δ is

$$\delta\phi = \psi, \quad \delta\bar{\psi} = B.$$

And

$$\Psi(\phi, \psi, \bar{\psi}, B) = \frac{1}{2}\int dt \bar{\psi}_i (g^{ij} B_j + 2i\partial_t \phi^i + g^{ij}\Gamma^k_{jl}\bar{\psi}_k \psi^l).$$

Integrating out B to find the following partition function of SQM

$$e_s(E) = \int d\phi d\psi d\bar{\psi} \frac{\hbar^{\dim(X)}}{\sqrt{g}} e^{-\frac{1}{\hbar}\int dt \left[\frac{1}{2} g_{ij}\partial_t\phi^i \partial_t \phi^j - i\bar{\psi}\nabla_t \psi - \frac{1}{4} R^{ij}_{kl}\bar{\psi}_i \bar{\psi}_j \psi^k \psi^l\right]}$$

(2) Consider the Kuranishi model of the two-term complex on $M = X$

$$TLX|_M \stackrel{ds}{\to} E|_M.$$

Show that the kernel is TX and cokernel is T^*X. Therefore the obstruction bundle is $ob = TX$. Excess intersection formula tells that the infinite dim integral for $e_s(E)$ is in fact computing

$$e_s(E) = e(ob) = Euler(X).$$

(3) Consider the limt $\hbar \to 0$ in $e_s(E)$. It turns out that only the part when $\phi, \psi, \bar{\psi}$ are constant will contribute to the integral (SUSY implies a cancellation on integrating out other Fourier modes). Show that the integration of constant modes in $e_s(E)$ gives

$$e_s(E) = \int_X Pf(R)$$

where Pf is the Pfaffian of the curvature 2-form R. This is the Chern-Gauss-Bonnet Theorem.