1. 4th Day: An example of the renormalization group flow

Consider a two-dimensional theory on $\mathbb{R}^2 = \mathbb{C}$ with fields chiral fermions

$$\psi \in \Omega^{1/2,0}(\mathbb{C}, V), \quad \psi' \in \Omega^{1/2,0}(\mathbb{C}, V^*) \tag{1}$$

and anti-chiral fermions

$$\overline{\psi} \in \Omega^{0,1/2}(\mathbb{C}, W) \quad \overline{\psi}' \in \Omega^{0,1/2}(\mathbb{C}, W^\vee) \tag{2}$$

Here V, W are complex vector spaces.

The Lagrangian is

$$\int \langle \psi, \overline{\partial}\psi' \rangle + \left\langle \overline{\psi}, \partial\overline{\psi}' \right\rangle + \sum \langle M_{V,i}\psi, \psi' \rangle \left\langle M_W^i \overline{\psi}, \overline{\psi}' \right\rangle \tag{3}$$

for some $M_{V,i} \in \mathrm{End}(V)$ and $M_W^i \in \mathrm{End}(W)$. The interaction only depends on $M = \sum M_{V,i} \otimes M_W^i \in \mathrm{End}(V \otimes W)$.

Question 1. *Calculate the propagator for the ψ-fields and the $\overline{\psi}$-fields.*

Question 2. *Show that there are one-loop counter-terms of the form*

$$\hbar \log(\epsilon) \left\{ \sum \langle (M_{V,i}M_{V,j})\psi, \psi' \rangle \left\langle M_W^i M_W^j \overline{\psi}, \overline{\psi}' \right\rangle \pm \langle (M_{V,j}M_{V,i})\psi, \psi' \rangle \left\langle M_W^i M_W^j \overline{\psi}, \overline{\psi}' \right\rangle \right\} \tag{4}$$

I don't remember which sign is correct in this formula. Also there may possibly be other one loop counter terms, here we are only interested in the ones with a coefficient of $\log \epsilon$.

As a hint, the one one-loop logarthimic divergence comes from diagrams with two vertices and two propagators, one of which is the $\psi - \psi'$ propagator and one of which is the $\overline{\psi} - \overline{\psi}'$ propagator. There are two possibilities for such diagrams, depending on the orientation of the propagators.

Question 3. *Conclude that, at one loop order, this theory is generally not scale invariant, and that the tensor $M \in \mathrm{End}(V) \otimes \mathrm{End}(W)$ defining the interaction flows.*