Tudor: Day 2

B-type (Dolbeault) quantum mechanics

The two relevant superfields for the 1d SUSY algebra are called chiral and fermi superfields (or chiral and fermi multiplets). In components, a chiral Φ and a fermi Γ contain

$$\begin{aligned} \Phi &= \phi + \tfrac{1}{2}\theta\chi + \theta\bar\theta\dot\phi\,, \qquad & \overline{D}\Phi = 0 \\ \Gamma &= \gamma + \theta g + \theta\bar\theta\dot\gamma\,, \qquad & \overline{D}\Gamma = 0 \end{aligned} \tag{1}$$

where the component fields ϕ, g are (complex) bosonic and $\chi, \dot\gamma$ are (complex) fermionic. In other words, $\Phi : \mathbb{R}\times\Pi\mathbb{C} \to \mathbb{C}$ and $\Gamma : \mathbb{R}\times\Pi\mathbb{C}\to\Pi\mathbb{C}$.

The SUSY transformations of component fields are

$$\begin{array}{c|cccc|cccc} & \phi & \chi & \bar\phi & \bar\chi & \gamma & g & \bar\gamma & \bar g \\ \hline Q & 2\chi & 0 & 0 & -\dot{\bar\phi} & g & 0 & 0 & 2\dot{\bar\gamma} \\ \overline{Q} & 0 & -\dot\phi & 2\bar\chi & 0 & 0 & -2\dot\gamma & -\bar g & 0 \end{array} \tag{2}$$

You may convince yourself that $[Q,\overline{Q}] = 2\partial_t$ on any component field. The basic action for a fermi and/or a chiral multiplet looks like

$$S = \int dt d\theta d\bar\theta \big[\tfrac{1}{2}\Phi^\dagger\partial_t\Phi - \Gamma^\dagger\Gamma\big] + \int dt d\theta\, \Gamma J(\Phi)\Big|_{\bar\theta=0} + \int dt d\bar\theta\, \Gamma^\dagger J(\Phi)^\dagger\Big|_{\theta=0}\,, \tag{3}$$

where J is a *holomorphic* function of chiral fields, and appears in a "fermionic superpotentials" or "J-term superpotentials." The function J contains the interactions. In component form, the action is

$$S = \int dt \Big[|\dot\phi|^2 - |g|^2 + gJ(\phi) + \bar g\overline{J(\phi)} + 2\bar\chi\dot\chi + 2\bar\gamma\dot\gamma + 2J'(\phi)\gamma\chi - 2\overline{J'(\phi)}\bar\gamma\bar\chi\Big] \tag{4}$$

1. Find the EOM for g, and eliminate it from the action.

2. Describe the Hilbert space $\mathcal{H}$ and the algebra Ops of polynomial local operators. Ignore factors of 2.

Hint: states in the Hilbert space can be described as functions $f(\phi,\bar\phi,\bar\chi,\gamma)$

You should find that, as vector spaces, we roughly have Ops $\approx \mathcal{H}\otimes\mathcal{H}$.

3. Consider the $\overline{Q}$-cohomology. The Noether charge associated to $\overline{Q}$ is

$$\mathcal{Q} = \dot\phi\bar\chi + J\gamma\,, \tag{5}$$

up to numerical factors. How does $\mathcal{Q}$ act on $\mathcal{H}$? Being a bit naughty and assuming that the bosonic part of the Hilbert space consists of polynomials in $\phi,\bar\phi$ (rather than L^2 normalizable functions), find $H^\bullet_{\mathcal{Q}}(\mathcal{H})$.

(This is a good local model. Globally, either the bosonic target will be compact, or we must introduce twisted mass deformations as in Mat's Friday talk, which have the effect of dressing polynomial wavefunctions with Gaussian decay factors.)

4. Consider the theory of just a single chiral multiplet Φ (no Γ). Find the $\mathcal{Q}$-cohomology of the Hilbert space (by specializing the answer above) AND the corresponding $\bar{Q}$-cohomology of polynomial local operators. Hint: on next page.

You should have found $H^\bullet_{\overline{Q}}(\mathrm{Ops}) \simeq \mathbb{C}[\phi, \partial/\partial\phi]$, acting on $H^\bullet_{\mathcal{Q}}(\mathcal{H}) \simeq \mathbb{C}[\phi]$.

Prior to taking cohomology, we had $\mathcal{H} \simeq \mathbb{C}[\phi, \bar{\phi}, \bar{\chi}] \simeq \Omega^{0,\bullet}(\mathbb{C})$, with $\mathcal{Q} = \bar{\partial}$ acting as the Dolbeault operator. Clearly the part of the operator algebra containing ϕ and $\partial/\partial\phi$ commutes with the action of the Dolbeault operator; thus the action of $\mathbb{C}[\phi, \partial/\partial\phi]$ descends to cohomology.

Global picture: with multiplet chiral multiplets, one can construct SQM whose target is a complex, Hermitian manifold $\mathcal{X}$. Chiral multiplets provide local holomorphic coordinates on $\mathcal{X}$. The Hilbert space will be $\Omega^{0,\bullet}(\mathcal{X})$, with $\mathcal{Q}$ acting as the Dolbeault operator. (There is also a quantum correction that twists by the square root of the canonical bundle, though it can be removed by hand if desired.) On top of this, one can add n fermi multiplets $\Gamma^1, ..., \Gamma^n$, which form a basis for the fiber of an odd holomorphic, Hermitian vector bundle $\mathcal{E} \to \mathcal{X}$. The Hilbert space becomes

$$\mathcal{H} = \Omega^{0,\bullet}(\mathcal{X}, \Lambda^\bullet \mathcal{E})\,. \tag{6}$$

Each Γ can also be given a J-term superpotential $\sum_i J_i \Gamma^i$. Geometrically, $j(\phi) = \sum_i J_i(\phi)\gamma^i$ is a holomorphic section of $\mathcal{E}$. Then the supercharge acts on $\mathcal{H}$ as a modified Dolbeault operator

$$\mathcal{Q} = \bar{\partial} + j\wedge\,, \tag{7}$$

where $j\wedge : \Omega^{0,i}(\mathcal{X}, \Lambda^j\mathcal{E}) \to \Omega^{0,i}(\mathcal{X}, \Lambda^{j+1}\mathcal{E})$ is exterior product with a section of $\mathcal{E}$.

Boundary conditions for the B-twist of 2d $\mathcal{N} = (2, 2)$

We'll focus on a local model: the 2d $\mathcal{N} = (2, 2)$ theory with target $\mathcal{M} = \mathbb{C}$, parameterized by a real superfield $\Phi_{2d} = \phi + ...$ The same setup was analyzed in class. In this case, however, we will focus on the B-twist. Recall that the 2d $\mathcal{N} = (2, 2)$ algebra is $[Q_+, \overline{Q}_+] = 2\partial_z$, $[Q_-, \overline{Q}_-] = 2\partial_{\bar{z}}$, where $z = t + is$ and $\bar{z} = t - is$. The B supercharge is

$$\overline{Q}_B = \overline{Q}_+ + \overline{Q}_-\,. \tag{8}$$

(Apologies for the bar – that's a nonstandard convention, used here to match the $\overline{Q}$ used as the Dolbeault operator in SQM.)

5. Suppose that we work on a half-space $\mathbb{R}_{s\leq 0} \times \mathbb{R}_t$. Find a second supercharge Q_B that, together with $\overline{Q}_B$, generates a 1d $\mathcal{N} = 2$ SUSY subalgebra, containing translations parallel to the boundary. In other words, find a Q_B that satisfies $[Q_B, \overline{Q}_B] = 2\partial_t$.

The 2d chiral superfield Φ_{2d} decomposes into a pair of superfields for the 1d algebra generated by $Q_B, \overline{Q}_B$: a 1d chiral $\Phi = \phi(s, t) + ...$ and a 1d fermi $\Gamma = \gamma + ...$ (where γ is a particular linear combination of the fermions in Φ_{2d}). It is straightforward to find this decomposition, but a little tedious. The action gets rewritten as

$$S = \int dt d\theta d\bar{\theta} \int ds \big[\tfrac{1}{2}\Phi^\dagger \partial_t \Phi - \Gamma^\dagger \Gamma\big] + \int dt d\theta \int ds\, \Gamma \partial_s \Phi\Big|_{\bar{\theta}=0} + \int dt d\bar{\theta}\, \Gamma^\dagger \partial_s \Phi^\dagger\Big|_{\theta=0}. \tag{9}$$

6. Suppose that the theory is on a half-space, and that we write the superpotential as

$$\Omega = \int_{\mathbb{R}_{s\leq 0}} ds\, \Gamma \partial_s \Phi\,. \tag{10}$$

Working in superspace, set $\delta\Omega = 0$, and find a boundary condition $\mathcal{B}_N$ naturally imposed by the boundary terms in the variation. What does this boundary condition do to ϕ? How about γ?

6b. Find the algebra of local operators on $\mathcal{B}_N$. To do this, it suffices to use a classical analysis. (The B-twist is nice like that.) Write down the possible polynomials that exist at the boundary, subject to the boundary condition from above, and modulo the 2d EOM. NOTE: the 2d EOM involve ∂_s derivatives! They look like $\ddot{\phi} = -\partial_s^2\phi$, $\dot{\chi} \sim \partial_s\bar{\gamma}$, $\dot{\bar{\chi}} \sim \partial_s\gamma$.... So you cannot quotient out by nearly as much as you did in a purely 1d system. Then take $\overline{Q} = \overline{Q}_B$ cohomology.

This is a somewhat nontrivial exercise. Once the dust clears, you should find that the polynomial local operators are $\mathbb{C}[\phi]$. This is HALF of the algebra that would have existed for a chiral field in 1d, which is $\mathbb{C}[\phi, \dot{\bar{\phi}}]$.

Mathematically, one *expects* the category of boundary conditions in the B-model with target $\mathbb{C}$ to be $D^bCoh(\mathbb{C})$. You have just found the structure sheaf $\mathcal{B}_N = \mathcal{O}_{\mathbb{C}}$.

6c. There is another way to find the local operators, but using a state-operator correspondence. Namely, End)($\mathcal{B}_N$) should be isomorphic to the ($\overline{Q}_B$-cohomology of the) Hilbert space on a strip $[0,1] \times \mathbb{R}_t$ with the boundary condition $\mathcal{B}_N$ on both sides. Squeeze the interval to zero size to get a purely 1d SQM. What is its target? What is the $\overline{Q} = \overline{Q}_B$ cohomology of its Hilbert space?

7. Similar: write the superpotential on a half-space as

$$\Omega = -\int_{\mathbb{R}_{s\leq 0}} ds\, \partial_s\Gamma\, \Phi\,. \tag{11}$$

Working in superspace, set $\delta\Omega = 0$, and find a boundary condition $\mathcal{B}_D$ naturally imposed by the boundary terms in the variation. What does this boundary condition do to ϕ? How about γ?

This second boundary condition $\mathcal{B}_D$ should be associated to the skyscraper sheaf at the origin, $\mathcal{O}_0$. In the derived category, the derived endomorphism algebra End($\mathcal{O}_0$) is an exterior algebra in one variable. This exterior algebra can be found fairly easily by repeating steps 6b or 6c above. It is the algebra $\mathbb{C}[\gamma]$ of local operators γ on the boundary.

Much more interesting coherent sheaves can be constructed as indicated in the lecture: add more 1d superfields at the boundary, and couple with J-terms... This produces nontrivial complexes.