1. 6th Day: An example of the OPE

Consider the same example as in the previous day, namely, a two-dimensional theory on $\mathbb{R}^2 = \mathbb{C}$ with fields chiral fermions

$$\psi \in \Omega^{1/2,0}(\mathbb{C}, V), \quad \psi' \in \Omega^{1/2,0}(\mathbb{C}, V^*) \tag{1}$$

and anti-chiral fermions

$$\overline{\psi} \in \Omega^{0,1/2}(\mathbb{C}, W) \quad \overline{\psi}' \in \Omega^{0,1/2}(\mathbb{C}, W^\vee) \tag{2}$$

Here V, W are complex vector spaces. Here we will choose a basis e_i of V, f_a of W, so that our chiral fermions are ψ_i, ψ^i and anti-chiral fermions are $\overline{\psi}_a$, $\overline{\psi}^a$. We have local operators which we call $\psi_i(z)$, etc, where we evaluate one of the fields at a point $z \in \mathbb{C}$. The Lagrangian is

$$\int \psi_i \overline{\partial} \psi^i + \int \overline{\psi}_a \partial \overline{\psi}^a + \int M^{ia}_{jb} \psi_i \psi^j \overline{\psi}_a \overline{\psi}^b. \tag{3}$$

We will compute the OPE of local operators, to leading order. This example is interesting, as it provides a very simple example of where we deform something well-understood mathematically – the tensor product of a vertex algebra and its complex conjugate – into something else. This example also allows us to understand the RG flow (seen above) using OPEs.

Question 1. *Verify that, in the free theory, where we have dropped the quartic interaction term, we have*

$$\begin{aligned} \psi_i(0)\psi^j(z) &\simeq \delta_i^j z^{-1} \\ \overline{\psi}_a(0)\overline{\psi}^b(z) &\simeq \delta_a^b \overline{z}^{-1} \end{aligned} \tag{4}$$

(Please ignore all factors of π, etc. when calculating this!)

Question 2. *Show that, when we turn on the interaction given by the tensor M^{ia}_{jb}, we get a new term in the OPE where*

$$\psi_i(0)\psi^j(z) \simeq M^{ja}_{ib} \overline{\psi}_a(0) \overline{\psi}^b(0) \frac{\overline{z}}{z} \log |z| \tag{5}$$

(Again, please ignore factors of π. The content is to find the log term. If $z = e^{2\pi \imath \theta}$, the factor of $e^{-4\pi\imath\theta}$ on the right must appear for symmetry reasons).

Question 3. *Conclude that the theory can not be scale invariant at the quantum level. That is, that there is no action of the group $\mathbb{R}_{>0}$ on the fields, lifting the action on the plane $\mathbb{R}^2 = \mathbb{C}$ by dilation, which preserves the one-loop OPE.*