1. 8th Day

Question 1. *Recall the identification* $\mathcal{O}(B\mathfrak{g}) = (C^\bullet(\mathfrak{g}), d_{CE})$*. Show that the DG Lie algebra of derivations of* $\mathcal{O}(B\mathfrak{g})$ *can be identified with the module* $C^\bullet(\mathfrak{g}, \mathfrak{g}[1])$ *where the Chevalley–Eilenberg differential is from the shifted adjoint action of* $\mathfrak{g}$ *on* $\mathfrak{g}[1]$*, realizing the space of vector fields as* $\mathrm{Vect}(B\mathfrak{g}) = C^\bullet(\mathfrak{g}, \mathfrak{g}[1])$*. One also has* $\Omega^1(B\mathfrak{g}) = C^\bullet(\mathfrak{g}, \mathfrak{g}^*[-1])$*.*

Question 2. *Consider a graded vector space* $\widetilde{\mathfrak{g}} = \mathbb{C}^n[-1] \oplus \mathbb{C}[-2]$*. We learned from 1st day exercises that equipping it with an* L_∞*-algebra structure is the same as giving a differential on* $\mathrm{Sym}^\bullet(\widetilde{\mathfrak{g}}^*[-1])$*; the resulting commutative differential graded algebra is the Chevalley–Eilenberg complex* $C^\bullet(\mathfrak{g})$ *for* $\mathfrak{g}$ *the resulting* L_∞*-algebra. Show that for* $\widetilde{\mathfrak{g}}$ *as above, such a structure is equivalent to a choice of function* $f\colon \mathbb{C}^n \to \mathbb{C}$ *with* $f(0) = 0$*, and that* $C^\bullet(\mathfrak{g})$ *is a resolution of the vanishing locus of* f*. [Hints: for* $n = 1$*:*

- $l_k\colon \widetilde{\mathfrak{g}}^{\otimes k} \to \widetilde{\mathfrak{g}}[2-k]$ *corresponds to* $d_k\colon \widetilde{\mathfrak{g}}^*[-1] \to \mathrm{Sym}^k(\widetilde{\mathfrak{g}}^*[-1])[1]$*.*
- *As we write* $C^\bullet(\widetilde{\mathfrak{g}}) = \mathbb{C}[[x]][\xi]$ *with* $|x| = 0$ *and* $|\xi| = -1$*,* $d_k\xi$ *is a multiple of* x^k*, say* $d_k\xi = a_k x^k$*.*
- *Indeed, one has* $d\xi = f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots \frac{f^{(k)}(0)}{k!}x^n + \cdots$*.]*

In general, the vanishing locus of a section of a trivial vector bundle of rank k *over an* n*-dimensional affine space is modeled by an* L_∞ *structure on* $\widetilde{\mathfrak{g}} = \mathbb{C}^n[-1] \oplus \mathbb{C}^k[-2]$*.*

Question 3. *As above, consider a graded vector space* $\widetilde{\mathfrak{g}} = \mathfrak{h} \oplus V[-1]$*. Given a Lie algebra structure on* $\mathfrak{h}$ *and an* $\mathfrak{h}$ *module structure on* V*, define an* L_∞ *structure on* $\widetilde{\mathfrak{g}}$ *such that* $C^\bullet(\mathfrak{g})$ *is a resolution of* $\mathcal{O}(V/\mathfrak{g})$*, the* $\mathfrak{g}$ *invariant functions on* V*. [Hint: The Chevalley–Eilenberg complex* $C^\bullet(\mathfrak{g}; M)$ *with coefficients in a* $\mathfrak{g}$*-module* M *computes the derived functor of taking Lie algebra invariants* $M \mapsto M^{\mathfrak{g}}$*.]*

Question 4. *Consider the Chern–Simons action functional*

$$S_{CS}(\alpha) = \int \frac{1}{2}\langle \alpha, d\alpha\rangle + \frac{1}{6}\langle \alpha, [\alpha, \alpha]\rangle$$

viewed as a function on $(\Omega^1_M \otimes \mathfrak{g})/(\Omega^0_M \otimes \mathfrak{g})$*. Combining the previous two exercises, verify that functions on the critical locus of* S_{CS} *are resolved by* $C^\bullet(\mathcal{L})$ *for* $\mathcal{L}$ *the DG Lie algebra* $\Omega^\bullet_M \otimes \mathfrak{g}$*.*

In general, given an L_∞*-algebra* $\mathcal{L}$ *equipped with a pairing* $\langle -, - \rangle : \mathcal{L}^{\otimes 2} \to \mathbb{C}[-3]$*, one can produce an action functional* $S(\alpha) = \sum_{k \geq 1} \langle \alpha, l_k(\alpha)\rangle$*, such that* $C^\bullet(\mathcal{L})$ *resolves the critical locus of* S*.*

Question 5. *Let $\mathfrak{g}$ be a vector space concentrated in cohomological degree* 0. *Show that there is an equivalence between Lie algebra structures on $\mathfrak{g}$ and Poisson structures on $\mathfrak{g}^*$ with bivector of homogeneous polynomial degree 1.*

Further, show that $\mathcal{O}(\mathfrak{g}^*)$ equipped with this Poisson bracket is the associated graded Poisson algebra of $\mathcal{U}(\mathfrak{g})$ with respect to the PBW filtration (the filtration by polynomial degree under the identification as vector spaces $\mathcal{U}(\mathfrak{g}) \cong \mathrm{Sym}^\bullet(\mathfrak{g})$).

Question 6. *Consider $\mathcal{E} = V \oplus V^*[-1]$ so that $\mathcal{O}(\mathcal{E}) = \mathbb{C}[x_i, \xi_i]$ and write $\mathcal{E} = B\mathfrak{g}$. Describe the map $\mathcal{O}(B\mathfrak{g})[-1] \to \mathrm{Vect}(B\mathfrak{g})$ given by $f \mapsto \{f, -\}$ in coordinates. Find a map $\mathcal{O}(B\mathfrak{g})[-1] \to \Omega^1(B\mathfrak{g})[-1]$ analogous to the de Rham differential, and describe it again in coordinates. Check that the formula for $\{f, -\}$ given above agrees with the expected answer from classical symplectic geometry, but using the* (-1)*-shifted symplectic structure.*

Question 7. *Consider n chiral free fermions on Σ where fields are $\Psi = (\Psi_i) \in \mathcal{E} = \Omega^{1/2,\bullet}(\Sigma; \mathbb{C}^n)$ and the action is $S = \frac{1}{2}\int \Psi\overline{\partial}\Psi = \frac{1}{2}\int \delta_{ij}\Psi_i\overline{\partial}\Psi_j$. We write $\mathcal{M} = \mathcal{E}[-1]$ with $d = \overline{\partial}$ and $l_k = 0$ for $k \geq 2$. Consider the natural action of $\mathcal{L} = \Omega^{0,\bullet}(\Sigma, \mathfrak{so}(n))$ on $\mathcal{M}$. Show that the corresponding action functional is $I^{\mathcal{L}}(\alpha, \Psi) = \frac{1}{2}\int \Psi\alpha\Psi$.*

Question 8. *In this exercise, you are asked to make a definition of β-functional as an obstruction to lifting the scaling symmetry. The point of the questions is to remind you of the important steps in the BV formalism; in particular, this is not asking you to do any actual computation.*

Suppose we are given a classical field theory $(\mathcal{E}, Q, \omega, I)$ on $\mathbb{R}^d$. Recall the scaling symmetry of $G = \mathbb{R}_{>0}$ on $\mathcal{E}$ is given by an action of $\mathbb{R}_{>0}$ on $\mathbb{R}^d$. We assume that the given classical theory is scale-invariant.

- *Show that $\mathcal{E}$ together with an action of $\mathfrak{g} = \mathbb{R}$ is a Maurer–Cartan element $I^{tot} = I + \epsilon I^{\mathbb{R}}$ of $(\mathcal{O}_{loc}(\mathcal{E})[-1] \oplus \epsilon\mathcal{O}_{loc}(\mathcal{E})[-1], Q, \{-,-\})$ with $|\epsilon| = 1$ or a local functional $I^{\mathbb{R}}$ of degree -1 such that $I^{\mathbb{R}} \in H^{-1}(\mathcal{O}_{loc}(\mathcal{E}), Q + \{I, -\})$.*
- *Fixing a renormalization scheme, there exists a pre-quantum field theory $\{\widetilde{I}[L]\}$ such that $\lim_{L\to 0}(\widetilde{I}[L] \text{ mod } \hbar) = I + \epsilon I^{\mathbb{R}}$. The anomaly for quantizing the classical scale symmetry is the obstruction of $\widetilde{I}[L]$ satisfying the scale L QME. Define the one-loop anomaly $\Theta^{(1)}[L]$ to be the obstruction modulo $\hbar^2$. Show that $\Theta^{(1)}[L]$ is closed under the BV differential.*
- *Assume that there is no obstruction to quantizing $\{I[L]\}$. Show that $\Theta^{(1)} = \lim_{L\to 0}\Theta^{(1)}[L]$ determines an element $H^0(\mathcal{O}_{loc}(\mathcal{E}), Q + \{I, -\})$. This is another definition of β-functional.*